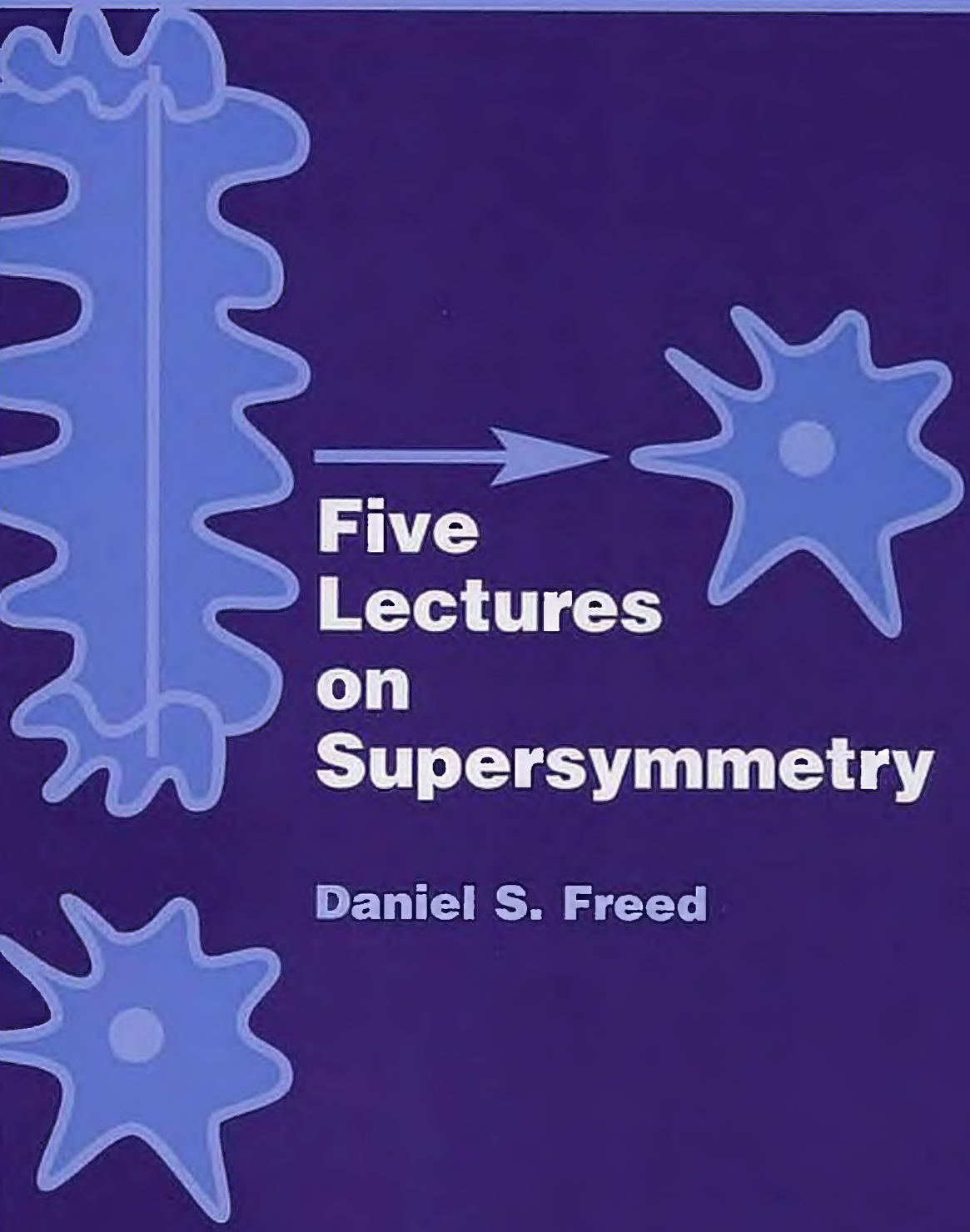

Five Lectures on Supersymmetry

Daniel S. Freed

American Mathematical Society

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ABSTRACT. These lectures aim to introduce the basic concepts of supersymmetry to mathematicians. The main goal is to expound classical supersymmetric field theories that are basic for applications in quantum mechanics and quantum field theory, providing the reader with enough background to explore quantum ideas.

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Preface

The five lectures recorded in this volume were first given in January, 1998 as part of the *Miniprogram in Geometry and Duality* at the Institute for Theoretical Physics in Santa Barbara. Shortly thereafter they were repeated at the Institute for Advanced Study (IAS) in Princeton. The aim of the lectures is to introduce basic concepts of supersymmetry to mathematicians. The beautiful applications of supersymmetry in mathematics all require ideas in quantum mechanics and quantum field theory which have heretofore proved difficult to render mathematically rigorous. By contrast, the mathematics which underlies these lectures is finite dimensional geometry and algebra. Our main goal is to expose the classical supersymmetric field theories which are the basis for the applications. Hopefully we provide the reader with enough background to explore the quantum ideas.

I learned much of this material during the 1996–1997 Special Year at the IAS, which is documented in the volumes *Quantum Fields and Strings: A Course for Mathematicians.* Some of the mathematical applications of supersymmetry mentioned above are discussed there. A more detailed treatment of most topics covered in these lectures may also be found in those books.

It is a pleasure to thank Pierre Deligne, with whom I wrote the detailed texts on which these lectures are based, and who contributed many comments about these notes. My understanding of this subject was shaped by many lectures and discussions. I single out Joseph Bernstein, David Kazhdan, John Morgan, Nathan Seiberg, Edward Witten. Sheldon Katz and Jeff Rabin made useful suggestions after reading a preliminary version, as did an anonymous referee. Finally, Arthur Greenspoon did his usual stellar job proofreading and making an index. I warmly thank them all.

I applaud the Institute for Theoretical Physics, in particular David Gross, for encouraging the interaction between mathematics and theoretical high energy physics. Thanks also to Karen Uhlenbeck for enlisting me to repeat these lectures in Princeton. My stay at the IAS, where these notes were written, was supported by the National Science Foundation and the J. Seward Johnson Sr. Charitable Trust. I am grateful for the support of these organizations.

Dan Freed
February, 1999

Introduction

Symmetry principles play a large role in both classical and quantum mechanics. In mechanics symmetries give rise to the conservation laws. Indeed, the conservation of energy and momentum in a classical system follow from invariance under basic motions of time and space. In a relativistic quantum system these motions are part of the Poincaré group, and elementary particles are classified according to its irreducible unitary representations of positive energy. Particles fall into two classes—bosons and fermions—and traditional symmetry principles do not mix the two classes. *Supersymmetries* are symmetries which exchange bosons and fermions. They were introduced into physics in the mid-'70s. A supersymmetric theory is more constrained than a nonsupersymmetric one, and in recent years that has led to great advances in the understanding of supersymmetric quantum field theories and string theories, particularly theories with a large amount of supersymmetry where the constraints are most stringent.

These advances in theoretical physics have had important mathematical consequences as well. Perhaps most spectacular was the advent of the Seiberg-Witten invariants in four dimensional geometry. These invariants contain all of the information encoded in Donaldson invariants, but are much easier to work with and exhibit many new features. The connection between the Seiberg-Witten invariants and Donaldson invariants has yet to be made mathematically rigorous, but in physics the precise link is derived by knowing the behavior of a certain supersymmetric gauge theory. Physics ideas played a particularly important role here since the mathematical ideas which developed independently of physics did not produce the Seiberg-Witten equations. In a different direction, new invariants arising from supersymmetric physics in two dimensions have led to advances in symplectic topology and enumerative algebraic geometry. Applications of supersymmetry to mathematics have impacted other parts of algebraic geometry, as well as differential geometry, representation theory, topology, and other fields. To be sure, there are mathematical ramifications of quantum field theory which do not involve supersymmetry, but it is safe to say that supersymmetry will continue to play an important role in the interaction with theoretical physics.

The five lectures recorded here are offered as background to understanding the physics behind these mathematical applications. We stop far short of explaining the quantum ideas. Rather, we restrict ourselves to material which is easily accessible—that is to say, provable—using present-day mathematics. In part, this is because an account of the quantum ideas aimed at mathematicians has just appeared (see References). But mostly we feel that the basic ideas of classical supersymmetric

field theory are unfamiliar to most mathematicians, and one cannot go deeply into the quantum theory without a firm grasp of these elementary ideas.

The first few lectures begin with some very general ideas, but then later lectures move on to more specific examples so that by the end the mathematical reader may feel lost in a zoo of unfriendly barely distinguishable rare beasts. We mathematicians love general concepts which we can apply to a wide range of examples. Think, for example, of groups, topological spaces, and vector bundles, just to name a few. Supersymmetric field theories are the antithesis: they are tied to specific dimensions, they depend on special properties of low dimensional Lie groups, and there are few examples. We have tried to emphasize the general ideas as much as possible, but a bit of taxonomy is inevitable. In a few places we give an expository overview of some physical ideas. Readers who have struggled through physics articles or seminars may benefit from these descriptions, but the reader unfamiliar with quantum field theory is well-advised to skip them at first reading.

Lecture 1, *What Are Fermions?*, explains a mathematical framework which includes bosons and fermions. While they exist quite democratically in the quantum world, the classical "odd" fields whose quantization give fermionic particles have a very different geometric significance than usual even fields. We adopt an algebro-geometric approach which we feel gives the right geometric intuition, allows the development of effective computational algorithms, and matches the existing physics literature. References to other approaches appear at the end of the lectures.

The space of states of a classical mechanical system is a symplectic manifold. Observables are functions on the state space, and via the symplectic structure an observable generates motion. For many important classical field theories that structure is summarized in a single expression, the lagrangian. In Lecture 2, *Lagrangians and Symmetries*, we explain how to pass from the lagrangian to the state space and how to construct the observables which correspond to symmetries. When we incorporate the odd fields of Lecture 1, the same framework applies to supersymmetries. The examples in this lecture are all taken from mechanics; only in subsequent lectures do we move on to fields.

Some general features of supersymmetric field theories are laid out in Lecture 3, *Supersymmetry in Various Dimensions*. We define the super Poincaré group and discuss its positive energy irreducible unitary representations, which are the particle multiplets of a supersymmetric quantum field theory. In this lecture we also survey the landscape of supersymmetric theories, introducing the reader to the specific features of different dimensions and different types of theories. At the end of the lecture we explain how to pass from free classical fields to the corresponding quantum particles.

In Lecture 4, *Theories with Two Supersymmetries*, we turn from a general description to an in-depth look at one particular example. Here we meet a familiar friend from geometry—the gradient flow equation—which is now reinterpreted in the context of supersymmetry.

We encounter more old geometric friends in Lecture 5, *Theories with More Supersymmetry*—Kähler manifolds and their quotients, hyperkähler manifolds and their quotients—and also make new friends—for example, special Kähler manifolds. This last lecture is aimed at the reader who aims to seriously study the physics. It highlights features of more complicated theories than those considered previously, and includes a detailed computation illustrative of the many which we omit.

As befits a series of lecture notes, there are no references in the main text and no comprehensive bibliography. Rather, we include a brief guide for further reading and encourage the reader to consult the bibliographies in those texts.

LECTURE 1
What Are Fermions?

Before we can talk about supersymmetry, which is a symmetry between bosons and fermions, we need to come to grips with the concept of a fermion. We begin in the quantum world by describing a particle as an irreducible representation of an appropriate symmetry group. Particles are classified as either bosons or fermions, and the distinction is neatly encoded by the mathematics of $\mathbb{Z}/2$-graded algebra. It is customary to use *super* in place of $\mathbb{Z}/2$-*graded* and so refer to this as *superalgebra*. We review the basic idea—the sign rule—and show how auxiliary odd parameters may be introduced as a convenient computational device. The mathematics of quantum fermions is completely standard: a very mildly noncommutative form of algebra.

In the second half of the lecture we recall how to pass from a classical bosonic particle to a quantum Hilbert space. (We will deal with fields in subsequent lectures.) The common saying is "Quantization is an art, not a functor" and in these lectures we have nothing to add to the vast literature on this subject. In fact, we will always quantize symplectic affine spaces, where there is a good mathematical theory. After reviewing the standard bosonic particle, whose quantization gives a quantum boson, we try to construct a classical model whose quantization also involves quantum fermions. Thus we are led to what we might call "classical fermions." These are the infamous anticommuting variables which give differential geometers the fits. The main point is to understand that if

$$\psi\colon \mathbb{R} \longrightarrow \mathbb{R}$$

is a compactly supported map, then the expression

$$\int_{\mathbb{R}} \psi(t)\dot{\psi}(t)\, dt$$

can be nonzero. At first glance this looks impossible, since the integrand appears to be the exact differential $d(\psi(t)^2/2)$. A physicist would respond, "ψ is anticommuting and your integration by parts has a sign error!" By the end of the lecture we will understand this response in mathematical terms and be able to nod agreeably.

The reader who is happy to work with anticommuting functions can skip this part of the lecture, but for the rest of us it is a bit of psychotherapy: after the mathematical explanation we can compute with these anticommuting variables and not worry about it!

Fermions in Quantum Mechanics

The state of a quantum mechanical system is represented by a vector (or line) in a *complex* Hilbert space. More concretely, it is a wave function and in a first course in quantum mechanics we learn that the complex values allow for the interference characteristic of quantum phenomena. Usually the wave function is normalized to have unit norm, and wave functions which differ by a constant phase represent the same physical state. We discuss both nonrelativistic and relativistic quantum mechanics.

In the nonrelativistic case a single particle in $\mathbb{R}^m$ is modeled by the Hilbert space

$$\mathcal{H} = L^2(\mathbb{R}^m, W),$$

where W is an irreducible complex representation of the rotation group $\mathrm{Spin}(m)$. (Recall that $\mathrm{Spin}(m)$ is the double cover of the rotation group $SO(m)$.) In the simplest case $m = 1$ and W is the trivial representation. Then $\mathcal{H}$ is the space of complex-valued L^2 functions on the line. It is the Hilbert space which underlies the free particle on the line or the harmonic oscillator, the first examples one usually encounters. In 3 space dimensions ($m = 3$) we might also take W to be the (complexification of the) standard 3-dimensional representation of the rotation group $SO(3)$. The corresponding particle is a massive spin 1 particle. If we take instead the 2-dimensional spin representation of $\mathrm{Spin}(3) \cong SU(2)$ then we obtain the Hilbert space of the massive spin 1/2 particle.

What do we mean by the "spin" of a representation of $\mathrm{Spin}(m)$? Suppose W is such a representation. Fix a 2-plane in $\mathbb{R}^m$ and consider the subgroup $SO(2) \subset SO(m)$ of rotations in that plane which fix the perpendicular plane. We work with the double cover $\mathrm{Spin}(2) \subset \mathrm{Spin}(m)$. Restricted to $\mathrm{Spin}(2)$ the representation W decomposes as a sum of one-dimensional complex representations, on each of which $\mathrm{Spin}(2)$ acts by $\lambda \mapsto \lambda^{2j}$, where $\lambda \in \mathrm{Spin}(2)$ and j is a half-integer. (We use half-integers so that the two-dimensional tautological representation of the $SO(2)$ subgroup of $SO(m)$ is the sum of the representations $j = 1$ and $j = -1$.) The spin of the representation W is the largest $|j|$ which occurs in the decomposition.

For example, the trivial representation has spin 0. It represents a *scalar particle*. The m-dimensional defining representation of $SO(m)$ has spin 1; the corresponding particle is sometimes called the *vector particle*. The reader can check that all exterior powers of this representation (except Λ^m) also have spin 1. The spin representations of $\mathrm{Spin}(m)$ have spin 1/2. One obtains higher spin by looking at the symmetric powers of the defining representation.

In nonrelativistic quantum mechanics one also has on the Hilbert space a representation of translations in space and a hamiltonian. Altogether we obtain a representation of (a central extension of) the Galilean group, but we will not need this. In the *relativistic* case the Galilean group is replaced by the Poincaré group, and we will encounter a "super" version of the Poincaré group in Lecture 3. So we take this opportunity to review some basics about representations of the Poincaré group.

Let $V = \mathbb{R}^{1,n-1}$ be a vector space with linear coordinates $t, x^1, x^2, \dots, x^{n-1}$ and Lorentz metric

$$g = c^2(dt)^2 - (dx^1)^2 - \cdots - (dx^{n-1})^2. \tag{1.1}$$

Here c is the speed of light, t represents time, and the x^i are the spatial coordinates. We often set

$$x^0 = ct$$

for convenience. In the nonrelativistic case $\mathbb{R}^m$ represents space, so we should think that the spacetime dimension n is equal to $m+1$. *Minkowski spacetime* M^n is the affine space which underlies V. It has a constant metric given by (1.1). At each point we have the *lightcone* of *lightlike* vectors of zero norm. Vectors inside the lightcone are *timelike* and have positive norm; vectors outside the lightcone are *spacelike* and have negative norm. (See Figure 1.)

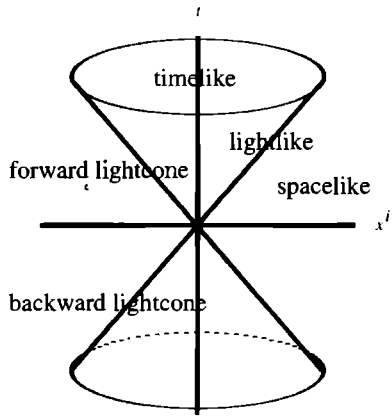

Figure 1: The Lorentzian vector space V

The group of isometries of V is the orthogonal group $O(1, n-1)$. It has four components (for $n \geq 2$) distinguished by the determinant (± 1) and whether or not the forward lightcone is preserved or is mapped to the backward lightcone. The double cover of the identity component $SO^+(1, n-1)$ is the Lorentz signature spin group $\mathrm{Spin}(1, n-1)$. The group of isometries of Minkowski spacetime M^n includes the subgroup of translations V, and the quotient by this subgroup is isomorphic to $O(1, n-1)$. The *Poincaré group* P^n is the double cover of the identity component of the group of isometries. It fits into the exact sequence

$$1 \to V \to P^n \to \mathrm{Spin}(1, n-1) \to 1. \tag{1.2}$$

Now a single relativistic quantum particle is defined to be an irreducible unitary positive energy representation $\mathcal{H}$ of P^n. Such representations were classified by Wigner long ago, and we quickly review the construction. First, we restrict the representation to the translation subgroup V. Since V is abelian, the representation decomposes as a direct sum (really direct integral) of one-dimensional representations on which V acts by a character

$$v \longmapsto \text{multiplication by } e^{i\alpha(v)/\hbar}, \qquad \alpha \in V^*.$$

The set of infinitesimal characters α which occur are permuted by the action of $SO^+(1, n-1)$ on V^*. Since the representation $\mathcal{H}$ is irreducible, these infinitesimal characters form an orbit of the action. There are two types of orbits which have positive energy. These are indicated in Figure 2. Notice that the axes are labeled E for energy and p_i for momentum. Energy and momentum are the dual variables to time and space, given the constant $\hbar$. Also, we have the relativistic mass formula

$$\frac{E^2}{c^4} - \sum_i \frac{p_i^2}{c^2} = m^2$$

for the mass square. The mass is constant on an orbit and so is an invariant of an irreducible representation. The two orbit types correspond to *massless* ($m = 0$) and *massive* ($m > 0$) representations.

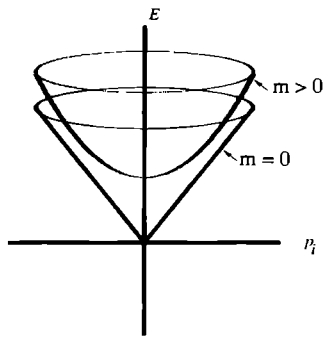

Figure 2: Orbits of $SO(1, n-1)$ in V^*, $n \geq 3$

Warning: In the two-dimensional case ($n = 2$) the massless orbit breaks up into two distinct orbits along the two rays of the positive lightcone, as indicated in Figure 3. We call them *right movers* and *left movers* since the corresponding characters are functions of $ct - x$ and $ct + x$ respectively. So there is a more refined classification of massless particles in two dimensions.

Now the representation $\mathcal{H}$ of P^n is obtained by constructing a homogeneous complex hermitian vector bundle over the orbit. More precisely, the total space of the bundle carries an action of $\mathrm{Spin}(1, n-1)$ covering the action of $SO^+(1, n-1)$ on the orbit. Such bundles may be constructed by fixing a point on the orbit and constructing a finite dimensional unitary representation of the stabilizer subgroup of that point, whose reductive part is called the *little group*. (Any finite dimensional representation factors through the reductive part.)

Consider first the massive case and fix the mass to be m. For convenience we set $c = 1$ and take as basepoint $(m, 0, \ldots, 0)$. Then the stabilizer subgroup, or little group, is easily seen to be $\mathrm{Spin}(n-1)$. Thus a massive particle corresponds to a representation of $\mathrm{Spin}(n-1)$, just as in the nonrelativistic case.

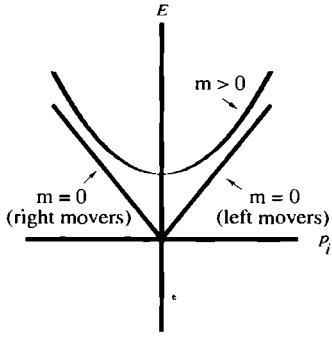

Figure 3: Orbits of $SO(1,1)$ in V^*

In the massless case we consider the basepoint $(1,1,0,\ldots,0)$. The stabilizer subgroup in this case is a double cover of the Euclidean group of orientation-preserving isometries of an $(n-2)$-dimensional Euclidean space. We can see this as follows. The group $SO^+(1,n-1)$ is the identity component of the group of conformal transformations of the $(n-2)$-dimensional sphere. We can view the sphere as the set of rays in the forward lightcone. Conformal transformations act on the forward lightcone, and the group of conformal transformations which fix a ray R in that sphere is isomorphic to the Euclidean group of the complement plus dilations. The action on points of R is given by the dilation factor at the corresponding point of the sphere, whence the claimed stabilizer subgroup. Massless particles correspond to finite dimensional unitary representations of this group, and such representations are necessarily trivial on the subgroup of translations. In other words, they factor through representations of the massless little group $\mathrm{Spin}(n-2)$.

We have the notion of spin of a representation as discussed earlier. In the massless case it is often termed *helicity*.

Given a homogeneous vector bundle, we take $\mathcal{H}$ to be the space of L^2 sections of the bundle over the orbit. (There is an $SO^+(1,n-1)$-invariant measure on the orbit.)

In either the massive or massless case the trivial representation of the little group corresponds to a spin 0 particle. A spin 1 particle corresponds to a representation of dimension $n-1$ in the massive case and $n-2$ in the massless case. Spin 1/2 particles correspond to spin representations of the little group; we discuss these more carefully in Lecture 3.

There are physical reasons why in interacting local quantum field theories one only sees massless particles of low spin. More precisely, in theories without gravity only massless particles of spin 0, spin 1/2, and spin 1 occur. Massless spin 1 particles only occur in *gauge theories*; a theory whose only massless particles have spin 0 and spin 1/2 is a *σ-model*. The *graviton* –the particle which mediates the gravitational force—is a massless particle of spin 2 and in theories of supergravity there are also massless particles of spin 3/2. That's it! There are no massless particles of higher spin in realistic theories.

Remark. In a local quantum field theory the CPT theorem states that the representation of P^n extends to a (projective) representation of the larger group whose quotient by translations is the entire group $SO(1, n-1)$. (Recall that in P^n we only consider the identity component of transformations which preserve the forward light cone.) Elements in the new component are represented by antilinear maps. The condition that the representation extend can be stated in terms of the little group; the precise statement depends on the parity of n. For n even it says that the representation of the little group is self-conjugate, i.e., either real or quaternionic. The statement for n odd is more complicated and we omit it.

What we have described so far is the Hilbert space of states of a single particle. Now we want to describe a two-particle state. If the individual particles are represented by Hilbert spaces $\mathcal{H}, \mathcal{H}'$, then the two-particle states are vectors in $\mathcal{H} \otimes \mathcal{H}'$. Suppose that the two particles are indistinguishable, so that $\mathcal{H} = \mathcal{H}'$. Then if v, v' are vectors in $\mathcal{H}$ representing one-particle states, the states $v \otimes v'$ and $v' \otimes v$ are identical. This is what we mean when we say that these are identical particles. But recall that identical states are represented by unit vectors which differ by a phase. So we have

$$v' \otimes v = \lambda\, v \otimes v' \tag{1.3}$$

for some phase λ. In other words, transposition of the factors leads to multiplication by λ. If we transpose twice we may suppose that we get back the original vector. In fact, there are more exotic possibilities in two dimensions, but we ignore them. Thus we will only consider $\lambda = 1$ or $\lambda = -1$. A particle is a *boson* if $\lambda = 1$ and a *fermion* if $\lambda = -1$.

To summarize: If $\mathcal{H}_B$ is the Hilbert space of states of a single boson B, then the symmetric square $\mathrm{Sym}^2\, \mathcal{H}_B$ is the space of states of two B particles. If $\mathcal{H}_F$ is the space of states of a single fermion F, then the exterior square $\bigwedge^2 \mathcal{H}_F$ is the space of states of two F particles.

There is a nice mathematical device to keep track of the difference between bosons and fermions. We discuss it in the next section. First a few remarks:

- The fact that a two-fermion state lives in the exterior square encodes the *Pauli exclusion principle*, which states that two identical fermions cannot be in the same state.
- In unitary local quantum field theories there is a connection between the spin of a particle and its *statistics*—whether it is a boson or a fermion. A particle of integral spin is a boson and a particle of half-integral spin is a fermion. This spin-statistics connection is often violated in nonunitary theories, and in particular in the topological field theories which have mathematical applications. In our discussions of field theory (starting in Lecture 3) we will always work in Minkowski spacetime and assume this spin-statistics connection. But it is violated in the superparticle example of Lecture 2.

Superalgebra

The idea of superalgebra, or $\mathbb{Z}/2$-graded algebra, is simple: Replace vector spaces by $\mathbb{Z}/2$-graded vector spaces, invoke the *sign rule* when commuting homogeneous elements, and assume that all structures (multiplications, etc.) are even.

Unless otherwise stated we use *graded* for $\mathbb{Z}/2$-*graded*.

A *graded vector space*

$$W = W^0 \oplus W^1$$

is simply a vector space which is presented as the direct sum of two subspaces. Elements in W^0 are termed *even*, and elements in W^1 are termed *odd*. We call the evenness or oddness of a homogeneous element its *parity* and use '$|a|$' to denote the parity of the homogeneous element a. Thus

$$|a| = \begin{cases} 0, & \text{if } a \in W^0; \\ 1, & \text{if } a \in W^1. \end{cases}$$

The *sign rule* states that in any equation where we permute the homogeneous elements a and b we pick up a sign $(-1)^{|a||b|}$.

We illustrate with two representative cases.

Example 1 (commutative superalgebra). An ordinary algebra A is a vector space with a multiplication $A \otimes A \to A$ which satisfies the usual associative and distributive laws. There is also an identity element. We say A is commutative if

$$ab = ba \tag{1.4}$$

for all elements $a, b \in A$.

To derive from this the definition of a *commutative superalgebra* we need to make the underlying vector space graded, postulate that the multiplication and identity are even, and use the sign rule. In detail: A is now a graded vector space

$$A = A^0 \oplus A^1.$$

The evenness of the identity element means $1 \in A^0$. The evenness of the multiplication is the statement

$$A^0 \cdot A^0 \subset A^0, \qquad A^0 \cdot A^1 \subset A^1, \qquad A^1 \cdot A^1 \subset A^0. \tag{1.5}$$

Finally, the sign rule applied to the commutative law (1.4) is the rule

$$ab = (-1)^{|a|\cdot|b|}\, ba \tag{1.6}$$

for homogeneous elements a, b. These are the axioms of a commutative superalgebra.

A common example of a commutative superalgebra is the exterior algebra of a vector space. It is actually $\mathbb{Z}$-graded, but we only remember the coarser $\mathbb{Z}/2$ grading in our current context. Equation (1.6) is the usual rule for the wedge product.

We will often consider the commutative superalgebra over $\mathbb{R}$ generated by odd elements $\eta_1, \ldots, \eta_N$. We denote it as

$$R = \mathbb{R}[\eta_1, \ldots, \eta_N]. \tag{1.7}$$

The generators η_i satisfy

$$\eta_i \eta_j = -\eta_j \eta_i. \tag{1.8}$$

In particular, the square of any η is zero.

Example 2 (super Lie algebra). An ordinary Lie algebra is a vector space $\mathfrak{g}$ with a bracket $[\cdot,\cdot]\colon \mathfrak{g}\otimes\mathfrak{g}\to\mathfrak{g}$ which is skew-symmetric and satisfies the Jacobi identity. Following our prescription we define a *super Lie algebra* to be a graded vector space

$$\mathfrak{g} = \mathfrak{g}^0 \oplus \mathfrak{g}^1$$

together with a bracket. The bracket is even, which means it satisfies conditions analogous to (1.5). The skew symmetry of the bracket is now

$$[a,b] = -(-1)^{|a|\,|b|}[b,a] \tag{1.9}$$

for homogeneous elements a, b. We state the Jacobi identity below.

These axioms are equivalent to the following: $\mathfrak{g}^0$ is an ordinary Lie algebra, $\mathfrak{g}^1$ is a representation of $\mathfrak{g}^0$, and there is a $\mathfrak{g}^0$-invariant *symmetric* pairing $\mathfrak{g}^1\otimes\mathfrak{g}^1\to\mathfrak{g}^0$ with $\big[a,[a,a]\big]=0$ for $a\in\mathfrak{g}^1$.

There is a super Lie algebra which one meets in differential geometry. Suppose ξ is a vector field on a manifold. Let $\mathrm{Lie}(\xi)$ denote the Lie derivative along ξ and $\iota(\xi)$ the contraction with ξ, both viewed as operators on differential forms. Then together with the exterior derivative d they generate a three-dimensional graded Lie algebra over $\mathbb{R}$, with $\mathrm{Lie}(\xi)$ even and $d, \iota(\xi)$ odd. The only nontrivial bracket is the Cartan formula

$$\mathrm{Lie}(\xi) = [d,\iota(\xi)] = d\iota(\xi) + \iota(\xi)d.$$

Another important construct is the following. If $W = W^0\oplus W^1$ is a graded vector space, then the algebra of endomorphisms $\mathrm{End}(W)$ is also graded. An endomorphism is even if it preserves W^0 and W^1; it is odd if it exchanges them. If we represent endomorphisms as block 2×2 matrices, then the even endomorphisms are diagonal matrices and the odd endomorphisms are off-diagonal matrices.

The Jacobi identity of a graded Lie algebra $\mathfrak{g}=\mathfrak{g}^0\oplus\mathfrak{g}^1$ is simply $\mathrm{ad}\big([a,b]\big) = \big[\mathrm{ad}(a),\mathrm{ad}(b)\big]$ for all elements $a,b\in\mathfrak{g}$.

Now there is a device we can use to simplify computations in superalgebra. Namely, we introduce auxiliary odd parameters $\eta_1,\dots,\eta_N$ which anticommute (1.8) and treat them as scalars. In other words, instead of working over $\mathbb{R}$, say, we work over the ring (see (1.7))

$$R = \mathbb{R}[\eta_1,\dots,\eta_N].$$

In this way we can work with even elements only and use the sign rule only when commuting these auxiliary parameters η_i past other elements. For example, we extend an even bilinear map $B\colon U\otimes V\to W$ to

$$B_R\colon (R\otimes U)\otimes(R\otimes V)\longrightarrow(R\otimes W)$$
$$B_R(ru,sv) = (-1)^{|u|\,|s|}\,rs\,B(u,v).$$

Here, as always, we use graded tensor products.

We illustrate this general description with the skew-symmetry law (1.9) in a graded Lie algebra $\mathfrak{g}=\mathfrak{g}^0\oplus\mathfrak{g}^1$. Thus suppose $a,b\in\mathfrak{g}^1$ are odd elements.

Then $\eta_1 a$ and $\eta_2 b$ are even elements. The skew-symmetry of the bracket (for even elements) says

$$[\eta_1 a, \eta_2 b] = -[\eta_2 b, \eta_1 a].$$

Now we simplify both sides, remembering that we are to treat the η_i as scalars. Thus on the left hand side we bring the η_1 to the left of the bracket and then commute η_2 past a and pick up a minus sign:

$$\text{LHS} = -\eta_1 \eta_2 [a, b].$$

On the right hand side we first commute η_1 past the even element $\eta_2 b$ (no sign!) and then bring the η_2 in front of the bracket:

$$\text{RHS} = -\eta_1 \eta_2 [b, a].$$

Combining the last three equations we learn that the bracket is symmetric on odd elements:

$$[a, b] = [b, a], \qquad a, b \text{ odd}.$$

A few remarks:

- The auxiliary odd parameters are introduced in this algebraic context purely as an aid in computation. They allow us to consider only even elements and so use the usual rules of whatever algebraic structure we have at hand. Experience shows that fewer sign mistakes are made this way.
- At a more formal level we can say that a graded Lie algebra $\mathfrak{g}$ is equivalent to a functor

  $$R \longrightarrow (R \otimes \mathfrak{g})^0$$

  from the category of commutative superalgebras to the category of ordinary Lie algebras. The tensor product here is a graded tensor product. Even elements of $R \otimes \mathfrak{g}$ are sums of products (even element of R) $\otimes$ (even element of $\mathfrak{g}$) and (odd element of R) $\otimes$ (odd element of $\mathfrak{g}$). We have a similar statement for any algebraic structure, not just Lie algebras.
- A variable, but finite, number of these "anticommuting constants" is always available. This matches physicists' use of such constants.

While on the subject of superalgebra, it is worth making a few remarks which we will need in later lectures. Our first two remarks are consequences of the sign rule and dictate conventions which are often not used in the physics literature. This is unfortunate, but since we have a consistent rule of signs in superalgebra it seems dangerous to violate it in special cases. The last remark is a choice of sign convention (among two valid possibilities) which we want to make explicit.

Remark 1 ($*$-operators). Suppose A is an ordinary algebra over $\mathbb{C}$. Then a *$*$-operation*

$$*\colon A \longrightarrow A$$

is a map which satisfies

$$\begin{aligned} (\lambda a)^* &= \bar{\lambda} a^*, && \lambda \in \mathbb{C}, \quad a \in A; \\ (a^*)^* &= a, && a \in A; \\ (ab)^* &= b^* a^*, && a, b \in A. \end{aligned} \tag{1.10}$$

Common examples are conjugation in the complex numbers, conjugation in the quaternions, and conjugate transpose in the algebra of complex square matrices. Now if A is a superalgebra over $\mathbb{C}$, then the sign rule demands that (1.10) be replaced by

$$(ab)^* = (-1)^{|a|\,|b|}\, b^* a^*, \qquad a, b \text{ homogeneous.}$$

In particular, in the commutative case we find that for all elements $a, b \in A$ we have

$$(ab)^* = a^* b^*, \qquad \text{if } A \text{ commutative.} \tag{1.11}$$

Note that the symmetrized product $ab + (-1)^{|a|\,|b|}\, ba$ of real elements a, b is real. (An element a is *real* if $a^* = a$.) The issue of this sign arises in Lecture 5 when we consider the four-dimensional superspacetime $M^{4|4}$.

Remark 2 (super Hilbert space). Our second remark concerns the definition of a super Hilbert space $\mathcal{H} = \mathcal{H}^0 \oplus \mathcal{H}^1$. The remark is algebraic, so we can restrict to finite dimensional Hilbert spaces. We take the inner product to be antilinear in the *first* variable. One of the axioms for the inner product in an ordinary Hilbert space describes how it behaves when the arguments are exchanged, and in the super situation this axiom picks up a sign:

$$\overline{\langle v, v' \rangle} = (-1)^{|v|\,|v'|} \langle v', v \rangle. \tag{1.12}$$

Here v, v' are homogeneous elements. Note that the evenness of the inner product implies that $\mathcal{H}^0$ and $\mathcal{H}^1$ are perpendicular. Now (1.12) implies that the norm square of an even element is real and the norm square of an odd element is purely imaginary. We take as positivity requirement the conditions

$$\begin{aligned} \langle v^0, v^0 \rangle &> 0, \\ -i\langle v^1, v^1 \rangle &> 0, \end{aligned} \tag{1.13}$$

where v^0 is even and v^1 is odd. We leave it to the reader to see from the sign rule that the eigenvalues of an odd self-adjoint operator T lie on the line $i^{1/2}\mathbb{R} \subset \mathbb{C}$, and the eigenvalues of an odd skew-adjoint operator lie on the line $i^{-1/2}\mathbb{R} \subset \mathbb{C}$. Furthermore, if T is odd skew-adjoint, then

$$i[T, T] \geq 0. \tag{1.14}$$

Remark 3 (super exterior algebra). Finally, we discuss a sign convention. Suppose $W = W^0 \oplus W^1$ is a super vector space with basis $\{e_i, f_\alpha\}$ and dual basis $\{e^i, f^\alpha\}$. Consider the exterior algebra $\bigwedge W^*$. For the even part there is no question that

$$e^i \wedge e^j = -e^j \wedge e^i$$

as usual. The sign comes since the e^i have *cohomological* degree 1. The ambiguity comes when we consider $e^i \wedge f^\alpha$. Since f^α is odd it has odd parity and it is also of odd cohomological degree. So should we consider the total degree to be even? More formally, each element of $\bigwedge W^*$ has both a *parity* p and a *cohomological degree* c. Our choice is the sign rule

$$\omega \wedge \omega' = (-1)^{pp' + cc'}\, \omega' \wedge \omega, \tag{1.15}$$

where ω has parity p and cohomological degree c and similarly for ω'. In other words, when computing in superalgebra we keep track of two signs: one for the (non-super) cohomological degree and one for the (super) parity. (The other valid choice is to use the total degree $p+c$ and so replace the sign in (1.15) with $(-1)^{(p+c)(p'+c')}$. It leads to consistent computations with different signs in the formulas.) Our choice has the consequence

$$\begin{aligned} e^i \wedge f^\alpha &= -f^\alpha \wedge e^i, \\ f^\alpha \wedge f^\beta &= f^\beta \wedge f^\alpha. \end{aligned}$$

(The other sign convention gives a plus sign in the first equation.)

A brief return to quantum particles

Recall that the one-particle Hilbert space of a particle can be considered bosonic or fermionic according to how it behaves when we have identical particles. Thus it is natural to collect together all of the particles in a theory and form the *one-particle Hilbert space* $\mathcal{H}$ of the theory as the direct sum of the Hilbert spaces of the individual particles in the theory. We write it as a graded space

$$\mathcal{H} = \mathcal{H}^0 \oplus \mathcal{H}^1$$

where the even part consists of bosons and the odd part consists of fermions. This has the nice consequence that the two-particle states are elements of the symmetric square

$$\mathrm{Sym}^2 \mathcal{H} \;\cong\; \mathrm{Sym}^2 \mathcal{H}^0 \;\oplus\; \mathcal{H}^0 \otimes \mathcal{H}^1 \;\oplus\; \mathrm{Sym}^2 \mathcal{H}^1.$$

The first summand consists of states with two bosons, the second states with one boson and one fermion, and the third states with two fermions. But remember the sign rule! In the symmetric square of the one-particle fermion space $\mathcal{H}^1$ we pick up a sign

$$v \otimes v' = -v' \otimes v, \qquad v, v' \in \mathcal{H}^1$$

from commuting the odd elements v, v'. This is precisely the sign λ in (1.3). Thus the sign rule of superalgebra encodes the statistics of a particle.

Now it is clear that the k-particle states are elements of $\mathrm{Sym}^k \mathcal{H}$. The *Fock space* of the theory is the sum of all the symmetric powers. Note that the *vacuum* lies in the 0th symmetric power, which is simply the trivial space $\mathrm{Sym}^0 \mathcal{H} \cong \mathbb{C}$. In a *free* quantum field theory this is the entire structure. The Poincaré group acts in the natural way on the Fock space $\mathrm{Sym}^\bullet \mathcal{H}$ induced from the representation on the one-particle Hilbert space $\mathcal{H}$. The Hilbert space of the theory is the Hilbert space completion of the Fock space. An *interacting* theory has more—an S-matrix which encodes the asymptotic result, from the far past to the far future, of interactions.

In a relativistic theory the Poincaré symmetries act as even transformations on $\mathcal{H}$. These symmetries map bosons to bosons and fermions to fermions. The same is true for the infinitesimal Poincaré symmetries. *Supersymmetries* at the quantum level are simply odd endomorphisms of $\mathcal{H}$. We will see a simple example at the classical level in Lecture 2. In Lecture 3 we give a general description of the quantum supersymmetry algebras.

Classical particles and their quantization

There is a fairly clean mathematical theory which passes from a *free* classical particle or field to a corresponding free quantum system. This involves quantizing symplectic vector spaces. In this first lecture we only consider classical particles, not classical fields. Our goal here is twofold. First, we want to briefly remind the reader of the crudest ideas of this free quantization. Second, we want to understand how to write a classical model whose quantization leads to fermions. Thus we begin with ordinary (bosonic) classical models.

There are four main ingredients in a classical field theory:

- A *spacetime* M
- A function space $\mathcal{F}$ of *fields* on spacetime
- A set of *classical equations of motion* whose solution space is a subset $\mathcal{M} \subset \mathcal{F}$
- A symplectic structure on $\mathcal{M}$.

In Lecture 2 we discuss more systematically the framework of *lagrangian field theory*. We will see how the classical equations and the symplectic structure are derived from a lagrangian, and how the lagrangian also encodes crucial information about the symmetries in the theory. For today we simply write down these things without deriving them from some more fundamental objects. We remark that the symplectic structure on $\mathcal{M}$ exists for spacetimes with a distinguished time direction. Lagrangian field theory makes sense on more general spacetimes, for example on curved Riemannian manifolds.

For n-dimensional spacetimes the fields are functions on an n-dimensional manifold. In this lecture we consider classical nonrelativistic particles which are simply functions of time. So our "spacetime" is just time ($\mathbb{R}$) and $n = 1$. The physical "space" is the target of the map.

Let $W^0 = \mathbb{R}^m$ denote space. (This W has nothing to do with the W's which appeared previously.) We take it to have a standard inner product, which we write as g_{ij} with respect to some fixed basis. The "fields" in this theory are paths in space:

$$\mathcal{F}^0 = \{x \colon \mathbb{R} \to W^0\}.$$

We consider a free nonrelativistic particle of mass m. It experiences no force and so Newton's law states that the classical equation of motion is that its acceleration vanish:

$$\ddot{x}(t) = 0.$$

We use the dot notation for time derivatives. The space $\mathcal{M}^0$ of solutions is the set of maps which have constant velocity. This is a vector subspace of $\mathcal{F}^0$. If we fix a time t_0 then we can identify such a map by the position and velocity at that time:

$$\begin{aligned} \mathcal{M}^0 &\longrightarrow W^0 \oplus W^0 \\ x &\longmapsto x(t_0) \oplus \dot{x}(t_0) \end{aligned} \tag{1.16}$$

Here we should view $W^0 \oplus W^0$ as the tangent bundle to W^0. The symplectic structure on this space can be written as

$$\omega = m\langle \delta\dot{x}(t_0) \wedge \delta x(t_0)\rangle,$$

where $\langle\cdot,\cdot\rangle$ is the inner product on W^0 and δ is the differential on $\mathcal{M}$. The symplectic structure is independent of t_0. The notation will become clearer in the next lecture. Using the isomorphism (1.16) we can write it as

$$\omega(v \oplus w, v' \oplus w') = m\{\langle w, v'\rangle - \langle w', v\rangle\}.$$

Note that each of the two summands W^0 in (1.16) is lagrangian.

We will see in the next lecture that the classical equations and the symplectic form can be derived from the *lagrangian density*

$$L^0 = \frac{m}{2}|\dot{x}(t)|^2\,|dt|.$$

Quantization assigns to the symplectic vector space $\mathcal{M}^0$ a complex Hilbert space $\mathcal{H}^0$. We would like this assignment to be as functorial as possible. Because quantum states have a phase ambiguity, we only expect to produce a projective Hilbert space.

Here are two possible interpretations of the functoriality wish:

- We might demand that the symmetries of the symplectic vector space be represented as unitary operators on the Hilbert space. More precisely, we expect a projective representation of the symplectic group. This is usually constructed by taking a splitting

  $$\mathcal{M}^0 \cong L \oplus L'$$

  into the sum of two lagrangian subspaces. With respect to this splitting (called a *polarization*) we define the Hilbert space to be a certain space of L^2 functions on L. With an eye towards the future we single out the dense subspace of polynomial functions, which is the symmetric algebra on the dual space:

  $$\mathcal{H}^0 \supset \mathrm{Sym}^\bullet(L^*) \otimes \mathbb{C}. \tag{1.17}$$

  Of course, this description depends on a choice of lagrangian splitting, but it turns out that the underlying projective space is independent of the splitting. An equivalent statement is that there is a projective unitary representation of the symplectic group, called the *metaplectic representation*, and the quantum Hilbert space is the underlying complex Hilbert space of that representation. This is a very satisfactory fulfillment of the functoriality wish. We usually take the lagrangian splitting to be the one in (1.16) and so describe the Hilbert space as the space of functions on W^0. This is the spin 0 nonrelativistic representation we described at the beginning of the lecture.
- Instead of focusing on the points of $\mathcal{M}^0$ and the points of $\mathcal{H}^0$, we might consider the functions on $\mathcal{M}^0$ and the operators on $\mathcal{H}^0$. In the usual way, the functions on a symplectic vector space comprise a Poisson algebra. In this case the Poisson bracket on linear functions is

  $$\{\dot{x}^i(t), x^j(t)\} = \frac{g^{ij}}{m},$$

where g^{ij} is the inverse metric. (We review Poisson brackets in Lecture 2.) From this perspective quantization should be a map which assigns to each real-valued function on $\mathcal{M}^0$ a self-adjoint operator $\hat{f}$ on $\mathcal{H}^0$. The naive functoriality wish in this case is that Poisson brackets be mapped into operator brackets. However, it is more realistic to introduce a parameter $\hbar$, work over formal power series in $\hbar$, and ask only that to first order in $\hbar$ Poisson brackets go over into operator brackets:

$$[\hat{f}_1, \hat{f}_2] = -i\hbar\{f_1, f_2\}\hat{} + O(\hbar^2). \tag{1.18}$$

For a symplectic vector space it turns out that the $O(\hbar^2)$ correction is not necessary if we restrict to the finite dimensional space of polynomials of degree ≤ 2.

Question: Is there something classical we can quantize to obtain a super Hilbert space? In other words, is there something classical we can quantize to see fermions at the quantum level?

We work backwards. We don't immediately specify the fields and equations, but simply postulate a space of solutions which looks promising. Our first guess is to replace the symplectic vector space $\mathcal{M}^0$ by a graded symplectic vector space. For simplicity we consider an odd symplectic vector space $\mathcal{M}^1$. Recall the sign rule! The skew symmetry of the symplectic form on $\mathcal{M}^1$ means that the form is symmetric on the underlying ordinary vector space. It turns out that we also need to impose a positivity condition, and we record the sign here:

$$\omega(v, v) < 0, \qquad v \in \mathcal{M}^1.$$

As on even symplectic vector spaces we have Poisson brackets, only on two odd elements the Poisson bracket is symmetric. The Poisson bracket of two linear functions is a constant function, and the positivity condition in the language of Poisson brackets is

$$\{\psi, \psi\} < 0, \qquad \psi \in \left(\mathcal{M}^1\right)^*. \tag{1.19}$$

Now we imitate our discussion of quantization. In the first approach to quantization we must start by polarizing $\mathcal{M}^1$, i.e., we want to write it as the sum of two maximal isotropic subspaces. But over the reals there are no proper isotropic subspaces. Thus we complexify and consider complex polarizations. Then we can write

$$\mathcal{M}^1 \otimes \mathbb{C} \cong L_{\mathbb{C}} \oplus \overline{L_{\mathbb{C}}}$$

as the sum of complex lagrangians, provided that the dimension of $\mathcal{M}^1$ is even. Assuming that, and following the previous (algebraic) prescription (1.17) we form the Hilbert space

$$\mathcal{H} = \mathrm{Sym}^\bullet(L_{\mathbb{C}}^*). \tag{1.20}$$

Since $L_{\mathbb{C}}$ is odd, this is a finite dimensional vector space (between friends: an exterior algebra). So in this case the entire Hilbert space has an algebraic description.

The reader may recognize this as a standard construction of the spin representation of an orthogonal group in even dimensions. Exactly parallel to the previous story we find that the underlying projective space is canonically independent of the polarization, i.e., the group of symplectic symmetries of the odd vector space $\mathcal{M}^1$ (between friends: an orthogonal group) acts projectively on $\mathcal{H}$. The $\mathbb{Z}/2$ grading is the usual splitting of spinors in even dimensions.

In the second approach we attempt to represent Poisson brackets by operators. If we choose a basis $\{\psi^i\}$ of linear functions on $\mathcal{M}^1$, then the Poisson brackets are

$$\{\psi^i, \psi^j\} = -\frac{g^{ij}}{m} \tag{1.21}$$

for some positive definite bilinear form g. (The positive constant m is put in analogously to the mass in the previous case.) We recognize this as the defining relation of a Clifford algebra, and the quantization problem is to find a $\mathbb{Z}/2$-graded complex Hilbert space on which this Clifford algebra acts by self-adjoint operators with brackets satisfying (1.18). The solution to this problem is again the spin representation (Clifford module). Recall, though, that the sign rule in superalgebra forced us into some unconventional conventions about graded Hilbert spaces and self-adjoint operators. These show up here.

Emboldened by our success we now try to construct the other elements of a classical theory—a function space $\mathcal{F}^1$ and an equation of motion which cuts $\mathcal{M}^1$ out of this function space. Here we make a very simple ansatz: Let $W^1 = \mathcal{M}^1$ be the odd vector space we have been considering. Then define $\mathcal{F}^1$ to be the function space

$$\mathcal{F}^1 = \{\psi\colon \mathbb{R} \to W^1\} \tag{1.22}$$

and take the classical equation of motion to be simply

$$\dot{\psi}(t) = 0.$$

In the next lecture we will see that this equation and the symplectic form can be derived from the lagrangian density

$$L^1 = \frac{m}{2}\langle \psi(t), \dot{\psi}(t)\rangle \, |dt|. \tag{1.23}$$

Here $\langle\cdot,\cdot\rangle$ is the symmetric bilinear form whose inverse appears in (1.21).

Be careful! Until the last paragraph we always considered odd vector spaces in a purely algebraic context, and there we just manipulate odd vectors the same way we manipulate even vectors. But in (1.22) we are considering paths in an odd vector space, and so for the first time we are attempting to do geometry on such a space. What does that mean? Certainly we have given no indication so far of what that means, much less ascribed a meaning to (1.23). We resolve these issues now.

Spaces with anticommuting functions

There are two basic approaches to the elementary theory of manifolds: we can focus on the points or dually we can focus on the functions. The latter point of

view is more adapted to algebraic geometry, and it is the approach we adopt here. So to describe the odd vector space W^1 as a manifold, we need to specify its ring of smooth functions. For an even vector space W^0 the ring of smooth functions contains the dense subset of polynomial functions:

$$C^\infty(W^0) \supset \mathrm{Sym}^\bullet\big((W^0)^*\big).$$

So it is natural to *define* the ring of functions on an odd vector space W^1 to be

$$C^\infty(W^1) := \mathrm{Sym}^\bullet\big((W^1)^*\big).$$

Since W^1 is odd this is a finite dimensional algebra, as we have seen many times by now. The use of 'C^∞' in this context is a bit funny, perhaps, but as a differential geometer I like to think by analogy with the situation for smooth manifolds.

Here is the crucial point: The ring of functions $C^\infty(W^1)$ contains a large subring of *nilpotent* elements. In fact,

$$C^\infty(W^1)/\text{nilpotents} \cong \mathbb{R}.$$

Notice that this quotient is simply the ring of functions on a point. This observation is crucial: *The fact that $C^\infty(W^1)$ is not a commutative algebra is a red herring; more important is the fact that it contains nilpotent elements.* Of course, the ring of functions on a smooth manifold contains no nonzero nilpotents. On the other hand, nilpotent functions are not uncommon in algebraic geometry and we can gain intuition by considering a simple example of a *commutative* ring with nilpotents.

We do not mean to say that the signs in the superalgebra $C^\infty(W^1)$ are not important. They are. But for the basic intuition of how to picture the space W^1 they are less essential than the nilpotency.

Now to the example. It is simply the case where $\dim W^1 = 1$, but we forget about the $\mathbb{Z}/2$-grading. Thus consider a space P whose ring of functions is

$$C^\infty(P) = \mathbb{R}[\epsilon]/(\epsilon^2 = 0).$$

(An algebraic geometer would write $P = \operatorname{Spec}\mathbb{R}[\epsilon]/(\epsilon^2)$.) Now we ask: What is a map $P \to M$ for M a smooth manifold? This is opposite to our problem, where we map a smooth manifold ($\mathbb{R}$) into a space with nilpotents (W^1), and we will come to that shortly. In later lectures we will also meet this case of maps from spaces with nilpotents into ordinary manifolds. In general, a smooth map between manifolds is equivalent to an algebra homomorphism on functions in the opposite direction. So a map $P \to M$ is given by a homomorphism

$$\begin{aligned} C^\infty(M) &\longrightarrow \mathbb{R}[\epsilon]/(\epsilon^2) \\ f &\longmapsto A(f) + B(f)\epsilon. \end{aligned} \tag{1.24}$$

The condition that this be an algebra homomorphism leads to two equations. The first states that $A\colon C^\infty(M) \to \mathbb{R}$ is an algebra homomorphism. The only such homomorphisms are evaluation at some point $m \in M$:

$$A(f) = f(m).$$

The second equation states that B is a derivation over functions, so is given as directional derivative in the direction of some tangent vector $\xi_m \in T_m M$:

$$B(f) = \xi_m f.$$

Therefore, maps (1.24) are in 1:1 correspondence with the tangent bundle of M.

So how might we picture P? It is in some sense an abstract tangent vector. We could represent it as an arrow (Figure 4). Or we might think of it as some sort of thickened point, a point together with a "nilpotent cloud" or "nilpotent fuzz" surrounding it. We should think that the nilpotent cloud has a definite shape, but in more complicated examples that would be hard to represent in detail pictorially. The reader may wish to consider the space whose ring of functions is $\mathbb{R}[\epsilon]/(\epsilon^n)$ for larger n, or the space whose ring of functions is $\mathbb{R}[\epsilon_1, \epsilon_2]/(\epsilon_1^2, \epsilon_2^2)$ with everything commutative. Geometrically, each of these examples is a point with some nilpotent cloud around it.

Figure 4: Two renditions of the space P

Now what about a map in the other direction, a map $M \to P$? Such a map is given by a homomorphism $\mathbb{R}[\epsilon]/(\epsilon^2) \to C^\infty(M)$. Now since the ring of functions on a smooth manifold has no nilpotents, the image of ϵ is necessarily zero. In other words, any map $M \to P$ simply maps the whole manifold M to a point, the geometric point in P.

We apply this discussion to our situation:

- As a manifold we picture the odd vector space W^1 as a point with a nilpotent cloud surrounding it. By contrast with the motivating example P, the cloud surrounding W^1 has anticommuting functions on it, but the geometric picture is essentially the same. Certainly the drawing is the same.
- Any map $\psi\colon \mathbb{R} \to W^1$ is identically zero!

This last conclusion is not what we want, since we'd like to give meaning to odd functions and use them in classical lagrangians. In other words, we'd like to be able to probe the nilpotent cloud in W^1.

Here we use another idea from algebraic geometry, Grothendieck's "functor of points". The idea is that an ordinary manifold M cannot probe the fuzz in a target, but we need to use manifolds with fuzz to probe the fuzz. In our situation we are studying maps $\mathbb{R} \to W^1$, and so we introduce nilpotents in the domain by studying families of maps parametrized by a space with nilpotents. Thus let B the space whose ring of functions is

$$C^\infty(B) = \mathbb{R}[\eta_1, \ldots, \eta_N], \qquad \eta_i \eta_j = -\eta_j \eta_i.$$

This is the commutative superalgebra we used before as a computational aid in superalgebra. In our current geometric situation we again introduce auxiliary parameters η_i, but now in the guise of a space B. Geometrically, B is a point with

a cloud of "superfuzz". We consider the product space $B \times \mathbb{R}$, which has a ring of functions

$$C^\infty(B \times \mathbb{R}) = \mathbb{R}[\eta_1, \dots, \eta_N] \otimes C^\infty(\mathbb{R}).$$

As before, we treat the η_i as constants; geometrically we are working with a family of spaces parametrized by the base B. (See Figure 5.)

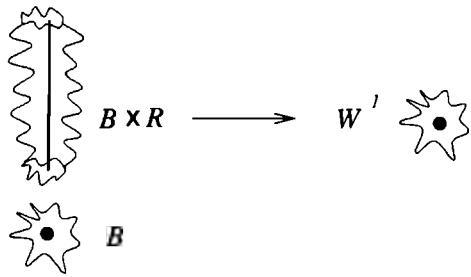

Figure 5: The family $B \times \mathbb{R}$ and a map to W^1

There are nontrivial maps $B \times \mathbb{R} \to W^1$. To illustrate let's start with a single η so that $C^\infty(B) = \mathbb{R}[\eta]$. Choose a basis $\{\psi^i\}$ of $(W^1)^*$; then

$$C^\infty(W^1) = \mathbb{R}[\psi^1, \dots, \psi^m], \qquad \psi^i\psi^j = -\psi^j\psi^i.$$

We suppose that the inverse metric in this basis is $g^{ij} = \langle \psi^i, \psi^j \rangle$. A map $\psi\colon B \times \mathbb{R} \to W^1$ is given as

$$\begin{aligned} C^\infty(W^1) &\longrightarrow \mathbb{R}[\eta] \otimes C^\infty(\mathbb{R}) \\ \psi^i &\longmapsto a^i(t)\eta \end{aligned} \tag{1.25}$$

for some functions $a^i \in C^\infty(\mathbb{R})$. Remember our postulate that when we introduce auxiliary parameters all of our structures are even. Here this means that the algebra homomorphism (1.25) is even, and so it maps the odd element ψ^i to an odd element of the codomain. With the odd parameter we have nonzero odd elements of the codomain, and so nonzero maps. We write

$$\psi^i(t) = a^i(t)\eta. \tag{1.26}$$

As advertised, ψ is a family of maps parametrized by B, and the formula makes clear that at the geometric point of B ($\eta = 0$) the map vanishes. This is what we found before we introduced the auxiliary odd parameters.

Finally, we can answer the question posed at the beginning of the lecture and ascribe meaning to (1.23). If we simply plug (1.26) into (1.23) we find that L^1 still vanishes. To get a nonvanishing answer we must go to a base space with two odd parameters η_1 and η_2. In that case we have

$$\psi^i(t) = a^i(t)\eta_1 + b^i(t)\eta_2,$$

and we compute

$$L^1 = \left\{ \frac{m}{2} g_{ij} \{ a^i(t)\dot{b}^j(t) - b^i(t)\dot{a}^j(t) \} \, dt \right\} \eta_1 \eta_2. \tag{1.27}$$

This is nonzero in the family, but as expected vanishes at the geometric point $\eta_1 = \eta_2 = 0$. The *action* S is the integral of the lagrangian, which here means the integral over the fibers of the projection $B \times \mathbb{R} \to B$:

$$S = \left\{ \int_{\mathbb{R}} \frac{m}{2} g_{ij} \{ a^i(t)\dot{b}^j(t) - b^i(t)\dot{a}^j(t) \} \, dt \right\} \eta_1 \eta_2.$$

(We assume our functions to have compact support.) Thus the action is a function on B, an element of $C^\infty(B)$. It is even, as we expect. (The lagrangian density (1.27) is even as well.) We might have made the same construction in the case of a classical bosonic particle $x\colon \mathbb{R} \to W^0$. In other words, we could have looked at a family of paths parametrized by an ordinary smooth manifold B, and we would have found the action to be a function on B. In that situation such families carry no more information than a single path. But when we have odd variables the families are essential to have nontrivial maps.

For fun the reader should write the analog of (1.27) with three auxiliary odd parameters. Now there is an $\eta_1\eta_2\eta_3$ term in $\psi^i(t)$ and the formulas are more complicated. The functoriality of these computations is clear: set $\eta_3 = 0$ to recover (1.27).

A few remarks:

- The auxiliary odd parameters were simply a computational device in superalgebra. In supergeometry they are essential for the geometric interpretation.
- Manifolds with superfuzz are more commonly known as *supermanifolds*, and we will of course refer to them as such.
- We will not make explicit the auxiliary odd parameters and the base spaces again in these lectures. The reader is well-advised to put them into some of the computations to better see the geometric pictures.
- We will not give a formal development of supergeometry and supermanifolds in these lectures. We will simply compute with them as we do in ordinary geometry. Thus we have tangent vectors, differential forms, bundles and connections, etc. We treat the nilpotent cloud as if it were a geometric direction. Hopefully the discussion here provides the necessary intuition and some idea as to how a more rigorous development would go. One imitates a development of ordinary differential geometry which emphasizes the functions rather than the points.
- The picture we have of "classical fermions" is in accord with the physical intuition. In some sense there are no classical fermions, which is the statement that the geometric points do not see the anticommuting functions. On the other hand we can picture them as living in a nilpotent cloud and so use our usual geometric ideas. When we pass to the algebraic situation of the quantum theory they are on an equal footing with the bosons.

From classical to quantum

The auxiliary odd parameters give a coherent picture of how we pass from a classical system with fermions to a quantum system with both bosons and fermions.

Namely, we work functorially over bases B whose ring of functions is a commutative superalgebra. In a given problem we fix a base B, then do all of the computations. Thus in a lagrangian field theory we define the space of fields, write the lagrangian, derive the equations of motion, find the classical space of solutions, compute the symplectic structure, etc. Then we apply some quantization procedure to obtain a Hilbert space with coefficient ring $C^\infty(B)$. So at the end of the day we produce a Hilbert space functorially from a commutative superalgebra. As we discussed earlier, this is equivalent to a graded Hilbert space.

LECTURE 2
Lagrangians and Symmetries

As we mentioned in Lecture 1, the geometry and physics of a classical field theory—for example, the equations of motion and the symplectic structure—can often be derived from a lagrangian density. That is true of all the theories we will meet. In this lecture we study some of the basic formal structure of *classical* lagrangian field theory, with particular emphasis on symmetries and the associated Noether currents. Although we develop the theory in some generality, our examples in this lecture are from mechanics. Examples with fields will occupy us in the remaining lectures. Our main example here—the superparticle—combines the two systems studied in Lecture 1.

The formal idea of a symmetry in a *quantum* theory is straightforward. Recall that the Hilbert space of the theory, as well as the one-particle Hilbert space of the theory, is graded

$$\mathcal{H} = \mathcal{H}^0 \oplus \mathcal{H}^1 \tag{2.1}$$

into bosons and fermions. A symmetry of the theory is a unitary operator on $\mathcal{H}$. In ordinary quantum mechanics we demand that a symmetry commute with the hamiltonian. The group of symmetries in a quantum field theory includes the Poincaré group and perhaps also a group of symmetries which commute with the action of the Poincaré group. These symmetries all preserve the grading in (2.1). We can also talk about the infinitesimal symmetries which generate one parameter groups of unitary transformations—physicists usually take them to be self-adjoint and often mathematicians prefer them to be skew-adjoint. *Supersymmetries* are operators which reverse the grading in (2.1). In quantum mechanics they commute with the hamiltonian; in fact, in the case we meet in this lecture the hamiltonian is the square of the supersymmetry. In higher dimensions they combine with ordinary infinitesimal symmetries to form an extension of the Poincaré algebra, which we discuss in Lecture 3. But in this lecture we focus on the classical story.

There are many different notations for calculus and differential geometry in finite dimensions, and even more for calculus on function spaces. I choose to write everything in terms of differential forms. The reader should try the computations in whatever notation is most comfortable.

There are some tricky signs in differential geometry which we encounter, and we comment on them at the end of the lecture. The presence of odd variables in classical theories with fermions is an additional sign danger. Finally, there are

some conventions and choices which must be made. We indicate some of these conventions as we go along. (We already met a few in Lecture 1; see (1.13), (1.18), and (1.19). These three choices are not independent.)

Some differential geometric preliminaries

So as not to interrupt the exposition later, we begin with some general facts from differential geometry and comment on some more sign conventions.

First, if M is a manifold and $E \to M$ a real or complex vector bundle with connection ∇, then we form the twisted de Rham "complex"

$$\Omega^0_M(E) \xrightarrow{d_\nabla = \nabla} \Omega^1_M(E) \xrightarrow{d_\nabla} \Omega^2_M(E) \xrightarrow{d_\nabla} \cdots . \tag{2.2}$$

It is not a complex in general, but rather

$$d_\nabla^2 = R_\nabla,$$

where R_∇ is the curvature of ∇.

Next, if $\phi\colon M \to X$ is a map between manifolds, then $d\phi \in \Omega^1_M(\phi^* TX)$. If now TX has a connection ∇, then

$$d_\nabla d\phi = \phi^* T_\nabla, \tag{2.3}$$

where T_∇ is the torsion of the connection ∇. We will apply this to the Levi-Civita connection of a Riemannian manifold; its torsion vanishes.

Recall that a *density* on a manifold M is a tensor field which in a local coordinate system $\{x^\mu\}$ is represented by

$$L(x)\,|dx^1 \cdots dx^n|.$$

It transforms under change of coordinates by the absolute value of the Jacobian of the coordinate change. In other words, it is a twisted n-form—the twisting is by the orientation bundle. We denote the set of densities by $\Omega^{|0|}(M)$. Then we define $\Omega^{|-p|}(M)$ to be the set of twisted $(n-p)$-forms. For example, an element of $\Omega^{|-1|}(M)$ in local coordinates looks like

$$j^\mu(x)\,\frac{\partial}{\partial x^\mu} \otimes |dx^1 \cdots dx^n|.$$

On the complex of twisted forms $\Omega^{|\bullet|}(M)$ we have the usual operations of exterior differentiation d, Lie derivative $\mathrm{Lie}(\xi)$ by a vector field ξ, and interior product $\iota(\xi)$, with the usual relationships among them. Just as we can integrate densities over manifolds, we can integrate $|-p|$-forms over codimension p submanifolds whose normal bundle is oriented. We comment shortly on our motivation for using twisted forms; little is lost by using ordinary forms instead.

Next, we make some brief remarks about signs and supergeometry. We define vector fields to act on functions by derivations and denote this action by writing the vector field ξ on the left of the function f in one of two ways: 'ξf' or '$\iota(\xi)df$'. If we want to move the vector field past the function and write '$df(\xi)$', then we need to invoke the sign rule. However, we will generally avoid this notation.

Given a graded vector space $W = W^0 \oplus W^1$, we define the *parity-reversed* vector space ΠW to be

$$(\Pi W)^0 = W^1, \qquad (\Pi W)^1 = W^0. \tag{2.4}$$

This has sign ambiguities when the coefficients are a general commutative superalgebra, so we are better off defining $\Pi W = W \otimes \mathbb{R}^{\mathrm{odd}}$, where $\mathbb{R}^{\mathrm{odd}}$ are the scalars viewed as an odd vector space. But for our purposes the heuristic definition (2.4) is adequate.

Finally, we remind the reader of our convention (1.15) for differential forms.

Lagrangian field theory

A disclaimer: The formal structure we develop here applies to many, but not all, examples. Modifications must be made to accommodate some topological terms (for example, Chern-Simons terms) and other types of terms involving p-form fields for $p \geq 2$. We will not encounter these subtleties. More importantly, our formalism does not apply to systems with certain types of constraints.

The stage for lagrangian field theory is set by a choice of *spacetime* M and a space of *fields* $\mathcal{F}$. Let n denote the dimension of spacetime.

First some remarks about M:

- In physics one often takes the spacetime to be Minkowski spacetime M^n, the affine space discussed following (1.1). In these lectures we work almost exclusively with this choice.
- In mechanics $n = 1$ and Minkowski spacetime M^1 is simply the affine space underlying $\mathbb{R}$; it represents time. This is the case for all examples considered in this lecture.
- For discussions of the *Hamiltonian* theory $M = M^1 \times N$ is time $\times$ space.
- In Euclidean field theory M is Euclidean space, or more generally a Riemannian manifold. This is often the case in applications to differential geometry.
- We allow M to be a supermanifold.

Formally, the space of fields $\mathcal{F}$ is the space of sections of a fiber bundle $E \to M$. We do not usually write fields in this form explicitly, but the theory described here is developed with this definition. For example, a *real scalar field* is a function $M \to \mathbb{R}$, which may be considered a section of the trivial bundle $M \times \mathbb{R} \to M$. A *connection* on a principal bundle $P \to M$ can be considered to be a section of a certain bundle of affine spaces. *Spinor fields* are sections of the vector bundle of spinors. In all cases there is a canonical evaluation map

$$\begin{aligned} \mathcal{F} \times M &\longrightarrow E \\ (\phi, x) &\longmapsto \phi(x). \end{aligned} \tag{2.5}$$

Lagrangians are usually defined in terms of this evaluation map and its derivatives, and we will simply use the standard notation '$\phi(x)$' or simply 'ϕ' for it. We allow E (and so $\mathcal{F}$) to be a supermanifold.

Our main examples. We consider two main examples in this lecture. Both examples are in mechanics ($n = 1$) where the spacetime $M = M^1$ is time. In both

examples we fix an external Riemannian manifold X which represents space. The reader may wish to consider the case when X is flat Euclidean space. The first example is a *classical particle*. The space of fields

$$\mathcal{F}_1 = \{x\colon M^1 \to X\}$$

is the space of paths on X. The second example is the *superparticle*, which in addition to the boson x has a fermionic partner ψ which is an odd tangent vector field along the image of x:

$$\mathcal{F}_2 = \{x\colon M^1 \to X, \quad \psi \in \Omega^0_{M^1}(x^*\Pi TX)\}.$$

Notice that $\mathcal{F}_1$ is an ordinary manifold whereas $\mathcal{F}_2$ is a supermanifold. The fermion ψ is interpreted by working over arbitrary commutative superalgebras (auxiliary odd parameters) as explained in Lecture 1.

We do calculus on $\mathcal{F} \times M$ and so study differential forms there. But instead of ordinary forms on the spacetime M we use twisted forms. Thus we work in the bigraded complex $\Omega^{\bullet,|\bullet|}(\mathcal{F} \times M)$: an element in $\Omega^{p,|q|}(\mathcal{F} \times M)$ is a p-form on $\mathcal{F}$ with values in $|-q|$-forms on M. The differential on M is denoted 'd' and the differential on $\mathcal{F}$ is denoted 'δ'. The total differential on $\mathcal{F} \times M$ is

$$D = \delta + d$$

and we have

$$D^2 = d^2 = \delta^2 = 0, \qquad d\delta = -\delta d.$$

We draw pictures of this complex as follows:

$$\begin{array}{c|ccc}
 & 0 & 1 & \cdots \\
\hline
|0| & & & \\
 & d\uparrow & & \\
|-1| & & \xrightarrow[\delta]{} & \\
 & & &
\end{array} \quad \mathcal{F}$$

$$M$$

There is an important subcomplex $\Omega^{\bullet,|\bullet|}_{\mathrm{loc}}(\mathcal{F} \times M)$ of *local* forms. The value of a form $\alpha \in \Omega^{\bullet,|\bullet|}(\mathcal{F} \times M)$ at a point $m \in M$ and a field $\phi \in \mathcal{L}$ on tangent vectors $\xi_1, \dots, \xi_p$ to $\mathcal{F}$ is a twisted form at m. The form α is *local* if this twisted form depends only on the k-jet at m of ϕ and the ξ_i. A theorem of F. Takens plays an important role in the theory.

Theorem 2.6 (Takens). *For $p > 0$ the complex $\left(\Omega^{p,|\bullet|}_{\mathrm{loc}}(\mathcal{F} \times M), d\right)$ of local differential forms is exact except in the top degree $|\bullet| = 0$.*

Finally we are ready to introduce the main character. A *lagrangian (density) L* is an element

$$L \in \Omega^{0,|0|}_{\mathrm{loc}}(\mathcal{F} \times M).$$

Thus for each field $\phi \in \mathcal{F}$, the lagrangian $L(\phi)$ is a density on M. Notice we have assumed that L is a *local* function of the field ϕ. That is certainly true for all examples we consider, but be warned that in quantum field theory one often introduces *effective lagrangians* which need not be local. (We contrast fundamental lagrangians and effective lagrangians in Lecture 5.) In fact, in these lectures all lagrangians L depend only on the *1-jet* of the fields. This means that the value of the density $L(\phi)$ at a point m of spacetime depends only on $\phi(m)$ and the first derivatives of ϕ at m. Certain features of classical field theory simplify for lagrangians which only depend on the 1-jet of the fields.

There are two reasons we take the lagrangian to be a density, rather than an n-form. First, it is notationally easier to consider it to have total degree 0. More importantly, writing it as a density makes clear the symmetries—there are often orientation-reversing symmetries of a theory. These are not present if writing the lagrangian necessitates a choice of orientation on M. We will see one example of this in Lecture 5.

The *classical action* S is the integral of the lagrangian density:

$$S(\phi) = \int_M L(\phi). \tag{2.7}$$

It is a real-valued function on $\mathcal{F}$. Of course, it is only well-defined on a subspace of fields (with finite action), though we are free to integrate over subsets of M. The action is a function on the space of fields $\mathcal{F}$ and is the main object of study in the calculus of variations. It remains off-stage in our presentation of the formal aspects –we express our computations locally on M in terms of the lagrangian.

The particle lagrangians. For the classical particle the lagrangian is

$$L_1 = \frac{1}{2}|\dot{x}|^2\, dt. \tag{2.8}$$

Usually we omit the argument 't' and simply write $L_1 = \frac{1}{2}|\dot{x}|^2\, dt$. Recall that '$x$' denotes the evaluation map (2.5). Also notice that we have fixed the standard orientation of the line and do not write the absolute values around dt. (The notation in some of the computations would be awful if we used twisted forms!) However, time-reversal *is* a symmetry of the theory. Also, since dt has cohomological degree 1 and $|dt|$ has cohomological degree 0, some signs in the computations differ after fixing an orientation.

The lagrangian of the superparticle is the sum of this lagrangian and a term which couples x to the fermion ψ:

$$L_2 = \left\{\frac{1}{2}|\dot{x}|^2 + \frac{1}{2}\langle\psi, \nabla_{\dot{x}}\psi\rangle\right\} dt. \tag{2.9}$$

The second term is essentially the lagrangian (1.23) discussed in Lecture 1 when X is flat space.

The supporting character in a classical field theory is the *variational 1-form*

$$\gamma \in \Omega^{1,|-1|}_{\mathrm{loc}}(\mathcal{F} \times M).$$

As we explain below, it encodes the usual integration by parts which one performs in the calculus of variations to compute the Euler-Lagrange equations. We define

the *total lagrangian* to be the sum of the lagrangian density and the variational 1-form:

$$\mathcal{L} = L + \gamma \in \Omega^{|0|}(\mathcal{F} \times M).$$

It has total degree 0. The defining property of a lagrangian field theory is that

$$(D\mathcal{L})^{1,|0|} = \delta L + d\gamma \in \Omega^{1,|0|}(\mathcal{F} \times M)$$

be *linear over functions* (on M). To explain this, note first that the tangent space at ϕ to the space of fields is

$$T_\phi \mathcal{F} = \Omega^0_M\big(\phi^* T(E/M)\big).$$

Here $\phi\colon M \to E$ is a section of the fiber bundle $E \to M$, and $T(E/M)$ is the tangent bundle along the fibers. Now $T_\phi\mathcal{F}$ is a module over Ω^0_M, the algebra of functions on M. A form $\beta \in \Omega^{1,|\bullet|}(\mathcal{F} \times M)$ is linear over functions if for all $(\phi, m) \in \mathcal{F} \times M$ we have

$$\beta_{(\phi,m)}(f\hat{\xi}) = f(m)\beta_{(\phi,m)}(\hat{\xi}), \qquad f \in \Omega^0_M, \quad \hat{\xi} \in T_\phi\mathcal{F}.$$

This condition is easier to understand in a computation, which we do immediately.

The variation of the lagrangian for the classical particle (2.8) is

$$\begin{aligned} \delta L_1 &= \langle \delta_\nabla \dot{x}, \dot{x}\rangle \wedge dt \\ &= \langle \delta_\nabla dx, \dot{x}\rangle \\ &= -\langle d_\nabla \delta x, \dot{x}\rangle \\ &= -d\langle \delta x, \dot{x}\rangle - \langle \delta x \wedge d_\nabla \dot{x}\rangle \\ &= -d\langle \dot{x}, \delta x\rangle - \langle \delta x, \nabla_{\dot{x}}\dot{x}\rangle \wedge dt. \end{aligned} \tag{2.10}$$

In the third line we use the fact that the Levi-Civita connection is torsion-free (see (2.3)), and twice we use $dx = \dot{x}\,dt$. Thus we *choose*

$$\gamma_1 = \langle \dot{x}, \delta x\rangle.$$

Then $\delta L_1 + d\gamma_1$ involves only δx and not $\delta_\nabla dx$. So it only involves variations of the fields, and not variations of derivatives of the fields. This is what the linearity over functions means. Notice that γ_1 is also linear over functions.

The manipulations (2.10) are standard and usually done under the integral sign in the classical action (2.7). In that context one only considers tangent vectors to $\mathcal{F}$ which have compact support, so that the integral converges. Then the integral of δL is simply the variation δS of the action, and it vanishes at critical points of S. The exact term in (2.10) integrates to zero. Thus for the classical particle the critical points of the action are the solutions to the geodesic equation

$$\nabla_{\dot{x}}\dot{x} = 0.$$

More generally, we define the space of *(classical) solutions* $\mathcal{M} \subset \mathcal{F}$ in any classical field theory to be the submanifold on which the *equation of motion* vanishes for all points of M:

$$(D\mathcal{L})^{1,|0|} = 0 \qquad \text{on } \mathcal{M} \times M.$$

Fields in $\mathcal{M}$ do not necessarily have finite action; rather, they are characterized as fields where the compactly supported variations of the action vanish.

Some terminology: Physicists refer to the space of classical solutions $\mathcal{M}$ as *on-shell* and the whole space of fields $\mathcal{F}$ as *off-shell*.

Now a few comments about the variational 1-form γ:

- If the lagrangian L depends only on the 1-jet of the fields, then there is a canonical choice for γ which is characterized as being linear over functions. We always choose this γ.
- If the lagrangian depends on higher derivatives, then there is more than one choice for γ, but for local lagrangians the difference between any two choices is d-exact (by Takens' Theorem 2.6).
- In many mechanics texts you will find this canonical γ written as '$p_i dq^i$'.

Next, we compute the variational 1-form and the equation of motion for the superparticle lagrangian (2.9). We already computed the variation of the first term, so now we compute

$$\begin{aligned} \delta\big(\frac{1}{2}\langle\psi, \nabla_{\dot{x}}\psi\rangle\, dt\big) &= \delta\big(\frac{1}{2}\langle\psi, d_\nabla\psi\rangle\big) \\ &= \frac{1}{2}\langle\delta_\nabla\psi \wedge d_\nabla\psi\rangle + \frac{1}{2}\langle\psi, \delta_\nabla d_\nabla\psi\rangle. \end{aligned} \tag{2.11}$$

The first term is linear over functions; we need to "integrate by parts" in the second:

$$\begin{aligned} \langle\psi, \delta_\nabla d_\nabla\psi\rangle &= -\langle\psi, d_\nabla\delta_\nabla\psi\rangle + \langle\psi, R(\delta x \wedge dx)\psi\rangle \\ &= -d\langle\psi, \delta_\nabla\psi\rangle + \langle d_\nabla\psi \wedge \delta_\nabla\psi\rangle + \langle\psi, R(\delta x \wedge dx)\psi\rangle \\ &= -d\langle\psi, \delta_\nabla\psi\rangle + \langle\delta_\nabla\psi \wedge d_\nabla\psi\rangle + \langle\delta x \wedge R(\psi,\psi)dx\rangle. \end{aligned} \tag{2.12}$$

Here R is the Riemann curvature tensor of X and we have used its symmetries in passing to the last equation. In the first step we used (2.2). Combining (2.10), (2.11), and (2.12) we find

$$\delta L_2 = -d\gamma_2 + \langle\delta x, -\nabla_{\dot{x}}\dot{x} + \frac{1}{2}R(\psi,\psi)\dot{x}\rangle \wedge dt + \langle\delta_\nabla\psi, \nabla_{\dot{x}}\psi\rangle \wedge dt, \tag{2.13}$$

where

$$\gamma_2 = \langle\dot{x}, \delta x\rangle + \frac{1}{2}\langle\psi, \delta_\nabla\psi\rangle.$$

We also read off from (2.13) the equations of motion:

$$\begin{aligned} \nabla_{\dot{x}}\dot{x} &= \frac{1}{2}R(\psi,\psi)\dot{x} \\ \nabla_{\dot{x}}\psi &= 0. \end{aligned} \tag{2.14}$$

Some remarks about these computations and results:

- The reader may want to put in some auxiliary odd parameters to understand the manipulations better. Also, there are many alternative ways to do

this computation (e.g. in local coordinates) and there is profit in trying alternative methods.

- Recall that with no auxiliary parameters the classical fermion field ψ vanishes, and so the superparticle reduces to the ordinary bosonic classical particle. In particular, the equations of motion reduce to the geodesic equation. I do not know any cogent geometric interpretation of the curvature term which appears in (2.14) when we put in auxiliary parameters (other than the general picture for such equations presented in Lecture 1).
- The lagrangian L_2 involves the square of the first derivative of the boson x but only the first power of the first derivative of the fermion ψ. This is typical for kinetic terms in lagrangians. If we introduce scaling dimensions $[x], [\psi]$ for the fields x, ψ, then the bosons and fermions will be weighted differently so that both terms have the same total weight. For example, we can assign

$$\begin{aligned}[x] &= -1/2\\ [\psi] &= 0\\ [t] &= -1.\end{aligned}$$

  Then both terms have total weight 0. Notice that no derivatives of ψ enter into the variational 1-form γ, and that the equation of motion for ψ is first order whereas the equation of motion of x is second order.
- The second term in the lagrangian couples x to ψ, and so the equations of motion also couple the two fields. By this argument alone we know that the geodesic equation gets modified. This is a version of Newton's law that for every action there is an equal and opposite reaction: the equation of motion for ψ depends on x, hence the equation of motion for x must depend on ψ.
- Assume that X is a *complete* Riemannian manifold. If we fix a time $t = t_0$, then a point in the space $\mathcal{M}_2$ of classical solutions is determined by the triple $\big(x(t_0), \dot{x}(t_0), \psi(t_0)\big)$. This is a point in the supermanifold described as the vector bundle

$$\pi^*\Pi TX \longrightarrow TX, \tag{2.15}$$

  where $\pi\colon TX \to X$ is the projection. In this supermanifold the ring of functions is the space of sections of the exterior algebra on the dual of the bundle (2.15). The space of classical solutions $\mathcal{M}_2$ is a sub-supermanifold of the supermanifold of fields $\mathcal{F}_2$, but the explicit description (2.15) depends on the choice of time.
- Below (2.16) we introduce a symplectic form on $\mathcal{M}_2$. We discuss the quantization of $\mathcal{M}_2$ briefly towards the end of the lecture.

Returning to a general classical field theory $\mathcal{L} = L + \gamma$, we define the *local symplectic form* ω to be

$$\omega := \delta\gamma \in \Omega^{2,|-1|}_{\mathrm{loc}}(\mathcal{F} \times M).$$

On-shell we have

$$\omega = D\mathcal{L} \qquad \text{on } \mathcal{M} \times M,$$

and so

$$D\omega = 0 \qquad \text{on } \mathcal{M} \times M.$$

Our data on-shell fits into the diagram:

$$\begin{array}{c|ccccc} & 0 & & 1 & & 2 \\ \hline |0| & L & \to & 0 & & \\ & & & \uparrow & & \\ |-1| & & & \gamma & \to & \omega \end{array} \quad \mathcal{M}$$

$$M$$

In mechanics ($n = 1$) for each time t the form $\omega(t)$ is a closed 2-form on the space of classical solutions $\mathcal{M}$, since $\delta\omega = 0$. It is independent of t, since $d\omega = 0$. A theory is *nondegenerate* if ω is a symplectic form.

In higher dimensions we must work in the Hamiltonian framework with a spacetime $M = M^1 \times N$ to see a symplectic structure. Then for each time $t \in M^1$ we integrate the local symplectic form to obtain a 2-form

$$\Omega(t) := \int_{\{t\}\times N} \omega$$

on the space of classical solutions. For simplicity we assume N compact so as not to have convergence issues. By Stokes' theorem we deduce that $\Omega(t)$ is independent of t. In good cases it is a symplectic form.

In classical mechanics texts the symplectic form ω is usually denoted '$dp_i \wedge dq^i$'.

We leave the reader to compute the symplectic forms

$$\omega_1 = \langle \delta_\nabla \dot{x} \wedge \delta x \rangle$$

for the classical particle and

$$\omega_2 = \langle \delta_\nabla \dot{x} \wedge \delta x \rangle + \frac{1}{2}\langle \delta_\nabla \psi \wedge \delta_\nabla \psi \rangle + \frac{1}{4}\langle \psi, R(\delta x \wedge \delta x)\psi \rangle \tag{2.16}$$

for the superparticle. (Here again we use the fact that the Levi-Civita connection is torsion-free: $\delta_\nabla \delta x = 0$.)

Symmetries and Noether currents

As with any mathematical structure we want to consider symmetries. We begin by recalling the notion of symmetry in symplectic geometry. Suppose that $(\mathcal{M}, \Omega)$ is a symplectic manifold. A symmetry (often called a *symplectomorphism*) is simply a diffeomorphism $\varphi\colon \mathcal{M} \to \mathcal{M}$ which preserves the symplectic form Ω. An infinitesimal symmetry, or hamiltonian vector field, is a vector field $\hat{\xi}$ with $\mathrm{Lie}(\hat{\xi})\Omega = 0$. In that case $-\iota(\hat{\xi})\Omega$ is a closed 1-form, and if it is exact it is the differential of a function Q_ξ which is the associated *momentum* or *charge*:

$$\delta Q_\xi = -\iota(\hat{\xi})\Omega. \tag{2.17}$$

(We use 'δ' for the differential on $\mathcal{M}$.) The charge is determined up to a locally constant function. If $\hat{\xi}_1, \hat{\xi}_2$ are infinitesimal symmetries with associated charges Q_1, Q_2, then the *Poisson bracket* $\{Q_1, Q_2\}$ is the function

$$\{Q_1, Q_2\} = \hat{\xi}_1 Q_2 = \iota(\hat{\xi}_2)\iota(\hat{\xi}_1)\Omega, \tag{2.18}$$

and it is a charge for the infinitesimal symmetry $[\hat{\xi}_1, \hat{\xi}_2]$. If $\mathfrak{h}$ is a Lie algebra of infinitesimal symmetries, then the Poisson algebra of all associated charges is a central extension of $\mathfrak{h}$ by locally constant functions.

In lagrangian field theory there is a local version of this story. We first define a symmetry, and it turns out there are two types: manifest and nonmanifest. To each symmetry we attach a local Noether current, and the global Noether charge is obtained in a Hamiltonian picture by integrating the Noether current over a spacelike codimension one submanifold. There is also a Poisson bracket on Noether currents which is a local version of the usual Poisson bracket on Noether charges, but we do not discuss it here. In mechanics there is no integration over space and the Noether current is the Noether charge. On-shell the global charge is related to the symmetry by (2.17)—it is the charge as defined in symplectic geometry. In lagrangian field theory there is a local version relating the Noether current to the local symplectic form, and there are explicit formulas for the currents and charges in terms of the lagrangian. In particular, to a manifest symmetry a lagrangian field theory attaches a *canonical* Noether charge, whereas in general a symplectic structure only determines a charge up to a locally constant function.

We begin by defining a symmetry in a lagrangian field theory. We work in our general setup with spacetime M, space of fields $\mathcal{F}$ (the space of sections of a fiber bundle $E \to M$), total lagrangian $\mathcal{L} = L + \gamma$, local symplectic form ω, and space of classical solutions $\mathcal{M}$. For simplicity we only give the definitions for infinitesimal symmetries, which are vector fields on $\mathcal{F} \times M$. First, we need to extend the notion of locality to vector fields. Formally, a vector field $\hat{\xi}$ on $\mathcal{F}$ is *local* if for some k the value of $\hat{\xi}_\phi \in \Gamma\big(M, \phi^* T(E/M)\big)$ at $m \in M$ depends only on the k-jet of ϕ at m. A vector field ξ on $\mathcal{F} \times M$ is said to be *decomposable and local* if it is the sum of a local vector field $\hat{\xi}$ on $\mathcal{F}$ and of a vector field η on M. Such vector fields preserve the bigrading on differential forms, and the Lie derivative by such vector fields commutes with both d and δ.

Definition. (i) A local vector field $\hat{\xi}$ on $\mathcal{F}$ is a *generalized infinitesimal symmetry* of L if there exists

$$\alpha_{\hat{\xi}} \in \Omega^{0,|-1|}_{\mathrm{loc}}(\mathcal{F} \times M)$$

such that

$$\mathrm{Lie}(\hat{\xi})L = d\alpha_{\hat{\xi}} \qquad \text{on } \mathcal{F} \times M \tag{2.19}$$

(ii) A decomposable and local vector field ξ on $\mathcal{F} \times M$ is a *manifest infinitesimal symmetry* if

$$\mathrm{Lie}(\xi)\mathcal{L} = 0 \qquad \text{on } \mathcal{F} \times M.$$

The definition of a manifest symmetry is the usual functorial notion—a symmetry of a mathematical structure is an automorphism which preserves all the data. A generalized symmetry is *nonmanifest* if we must choose $\alpha_{\hat{\xi}}$ in (2.19) nonzero; it only preserves the lagrangian up to an exact term.

We illustrate the definition in examples soon, but first make several remarks:

- There is ambiguity in the definition of $\alpha_{\hat{\xi}}$ for a nonmanifest symmetry—we can add any d-closed form. We must choose some $\alpha_{\hat{\xi}}$ to define the Noether current (see (2.21) below). The difference between any two choices for $\alpha_{\hat{\xi}}$

is exact, by Theorem 2.6. By contrast, the Noether current of a manifest symmetry is canonically determined.

- In the definition of a generalized infinitesimal symmetry we have said nothing about $\mathrm{Lie}(\hat{\xi})\gamma$. In fact, an argument based on Theorem 2.6 and its close cousins implies that for some $\beta_{\hat{\xi}} \in \Omega^{1,|-2|}_{\mathrm{loc}}(\mathcal{F} \times M)$, the identity

$$\mathrm{Lie}(\hat{\xi})\gamma = \delta\alpha_{\hat{\xi}} + d\beta_{\hat{\xi}} \qquad \text{on } \mathcal{M} \times M \tag{2.20}$$

holds on shell. We use this identity below to see that the Noether current is a local version of the conserved charge in symplectic geometry.

- In the manifest case it is awkward to have to compute that $\mathrm{Lie}(\xi)\gamma = 0$. In good cases this follows from $\mathrm{Lie}(\xi)L = 0$, which is much easier to check. Namely, suppose each element in a one-parameter group of manifest symmetries is induced from a bundle automorphism of $E \to M$ which covers a diffeomorphism of M. If the lagrangian L depends only on the 1-jet of the fields, and if it is preserved by the diffeomorphism of $\mathcal{F} \times M$, then it follows that γ is also preserved and we have a manifest symmetry.

For a generalized infinitesimal symmetry, we summarize (2.19) and (2.20) in the diagrams:

$$\begin{array}{c|c} & 0 \\ \hline |0| & \mathrm{Lie}(\hat{\xi})L \\ & \uparrow \\ |-1| & \alpha_{\hat{\xi}} \end{array} \quad \mathcal{F}, \qquad M$$

$$\begin{array}{c|ccc} & 0 & & 1 \\ \hline |0| & \mathrm{Lie}(\hat{\xi})L & & \\ & \uparrow & & \\ |-1| & \alpha_{\hat{\xi}} & \to & \mathrm{Lie}(\hat{\xi})\gamma \\ & & & \uparrow \\ |-2| & & & \beta_{\hat{\xi}} \end{array} \quad \mathcal{M}, \qquad M$$

It is easy to check that for a generalized infinitesimal symmetry $\hat{\xi}$, we have

$$\mathrm{Lie}(\hat{\xi})\omega = d(-\delta\beta_{\hat{\xi}}) \qquad \text{in } \Omega^{2,|-1|}(\mathcal{M} \times M).$$

This is a local version of the statement that $\hat{\xi}$ is a hamiltonian vector field. For a manifest infinitesimal symmetry ξ, the local symplectic form is preserved on the nose: $\mathrm{Lie}(\xi)\omega = 0$.

Let $\xi = \hat{\xi} + \eta$ be a decomposable and local vector field on $\mathcal{F} \times M$ which is either a manifest infinitesimal symmetry or a generalized infinitesimal symmetry with associated $\alpha_\xi \in \Omega^{0,|-1|}_{\mathrm{loc}}(\mathcal{F} \times M)$. (In the latter case $\eta = 0$, though we could generalize to allow η nonzero.) For a manifest infinitesimal symmetry set $\alpha_\xi = 0$. Define the *Noether current* to be

$$j_\xi := \left[\iota(\xi)\mathcal{L}\right]^{0,|-1|} - \alpha_\xi \in \Omega^{0,|-1|}_{\mathrm{loc}}(\mathcal{F} \times M). \tag{2.21}$$

In the manifest case the right hand side is simply the first term. It is easy to verify from (2.19), (2.20), and the Cartan formula that in all cases we have on-shell

$$dj_\xi = 0 \qquad \text{on } \mathcal{M} \times M, \tag{2.22}$$

$$\delta j_\xi = \left[-\iota(\xi)\omega\right]^{1,|-1|} + d\left(\beta^{2,|-2|}\right) \qquad \text{on } \mathcal{M} \times M, \tag{2.23}$$

for some $\beta^{2,|-2|} \in \Omega^{2,|-2|}_{\mathrm{loc}}(\mathcal{F} \times M)$. Equation (2.22) is the assertion that the Noether current j_ξ is *conserved*. Equation (2.23) is the local version of the correspondence (2.17) in symplectic geometry between an infinitesimal symmetry and its associated charge. In a Hamiltonian situation $M = M^1 \times N$ we define the Noether charge to be the function on $\mathcal{M}$ given by

$$Q_\xi = \int_{\{t\}\times N} j_\xi.$$

Of course, we restrict to field configurations on which the integral converges. The local conservation law (2.22) implies that Q_ξ is well-defined independent of t, i.e., that it is a constant of motion. The integral of (2.23) is exactly the relation (2.17) between the charge, the symmetry, and the symplectic form.

In mechanics the manifold N is a point and there is no integration. So $Q_\xi = j_\xi$.

Examples of symmetries in mechanics

We give a series of examples culminating in an example of an odd symmetry—a supersymmetry—for the superparticle.

Example 1 (linear momentum). Consider a classical free particle $x\colon M^1 \to X$ moving in flat space $X = \mathbb{R}^m$. We include the mass m in the lagrangian

$$L = \frac{m}{2}\sum_i (\dot{x}^i)^2\, dt.$$

Here we use standard coordinates on $\mathbb{R}^m$. As we computed before,

$$\gamma = m\sum_i \dot{x}^i\, \delta x^i.$$

We take as symmetry ξ_j translation in X in the jth coordinate direction:

$$\begin{aligned} \iota(\xi_j)dt &= 0 \\ \iota(\xi_j)\delta x^i &= \delta^i_j, \end{aligned} \tag{2.24}$$

where δ^i_j has its usual meaning. The reader should check that this is a manifest symmetry. This can be done by computation or by a simple geometrical argument. The associated current, or charge, is a component of the linear momentum (mass times velocity):

$$j_{\xi_j} = m\delta_{jk}\dot{x}^k.$$

Angular momentum can be treated similarly; it is the conserved charge associated to rotations.

We remark that the notation in (2.24) is not the usual one in the physics literature, nor in the mathematics literature for that matter! Usually one writes simply '$\delta x^i = \delta^i_j$' or perhaps '$\delta_{\xi_j} x^i = \delta^i_j$'. In our notation δ is the differential on the space of fields, whereas in these other notations δ is a vector field on that space. We choose our notation simply to be able to use the usual formulas for differential forms, and we live with the somewhat burdensome (2.24). We are happy to use the alternative equations $\xi_j t = 0, \xi_j x^i = \delta^i_j$ here, but in other examples (e.g. Example 2 below) this notation is not adequate.

Example 2 (time translation as a manifest symmetry). Again we consider the classical particle $x\colon M^1 \to X$, and now we allow an arbitrary Riemannian manifold X. We also introduce a (time-independent) potential function $V\colon X \to \mathbb{R}$ and the lagrangian

$$L = \Big\{\frac{m}{2}|\dot{x}|^2 - V(x)\Big\}\,dt. \tag{2.25}$$

This lagrangian is kinetic energy *minus* potential energy, the standard form for a particle lagrangian. The potential term does not affect the computation of γ, since it involves no derivatives of the fields so does not enter into any integration by parts. Time translation is a manifest symmetry of this theory: we translate in time and simultaneously pull back the field x by the inverse. This transformation fixes the evaluation map (2.5) and also the density dt, and so L is manifestly invariant. The corresponding infinitesimal symmetry is the vector field ξ on $\mathcal{F} \times M$ given as

$$\begin{aligned} \iota(\xi)dt &= 1 \\ \iota(\xi)\delta x &= -\dot{x}. \end{aligned} \tag{2.26}$$

The associated charge has contributions from both $\iota(\xi)\gamma$ and $\iota(\xi)L$. It turns out to be

$$j_\xi = -\Big\{\frac{m}{2}|\dot{x}|^2 + V(x)\Big\},$$

which is *minus* the *hamiltonian* or *energy*

$$H = -j_\xi = \frac{m}{2}|\dot{x}|^2 + V(x).$$

Notice the sign change from the lagrangian: The hamiltonian is kinetic energy *plus* potential energy. It is unfortunate that the hamiltonian is *minus* the Noether charge of time translation, but there is nothing we can do about it.

Example 2′ (time translation as a nonmanifest symmetry). Time translation is manifest because the action on fields is induced from the action on spacetime. We can make it nonmanifest simply by forgetting the action on spacetime. Let $\hat{\xi}$ be the vector field

$$\iota(\hat{\xi})\delta x = -\dot{x}$$

on $\mathcal{F}$. Then $\hat{\xi}$ is a nonmanifest symmetry of the lagrangian (2.25) with

$$\alpha_{\hat{\xi}} = -\big(\frac{m}{2}|\dot{x}|^2 - V(x)\big) + C.$$

We put in a constant C to emphasize the indeterminacy which exists for nonmanifest symmetries. The Noether current is

$$j_{\hat{\xi}} = \iota(\hat{\xi})\gamma - \alpha_{\hat{\xi}} = -\big(\frac{m}{2}|\dot{x}|^2 + V(x) + C\big).$$

Of course we have the same result as before, but now there is an undetermined constant in the hamiltonian.

Example 3 (system of identical harmonic oscillators). Here we take space X to be a real inner product space of dimension m and the potential to be quadratic:

$$L = \left(\frac{1}{2}|\dot{x}|^2 - \frac{1}{2}|x|^2\right) dt. \tag{2.27}$$

The equation of motion is $\ddot{x} = -x$, which leads to oscillatory motion. Consider the vector field $\hat{\xi}$ on $\mathcal{F}$ defined by

$$\iota(\hat{\xi})\delta x = Ax + B\dot{x},$$

where A is skew-symmetric and B is symmetric. Then $\hat{\xi}$ is a nonmanifest symmetry with

$$\alpha_{\hat{\xi}} = \frac{1}{2}\langle \dot{x}, B\dot{x}\rangle - \frac{1}{2}\langle x, Bx\rangle.$$

The associated charge is

$$j_{\hat{\xi}} = \langle \dot{x}, Ax\rangle + \frac{1}{2}\langle \dot{x}, B\dot{x}\rangle + \frac{1}{2}\langle x, Bx\rangle.$$

We make several remarks:

- For $B = 0$ the symmetry is manifest. These infinitesimal symmetries generate the action of the orthogonal group on X, and the lagrangian (2.27) is manifestly invariant under orthogonal transformations of the target. The associated charges in this case are angular momenta.
- For $A = 0$, $B = -\mathrm{id}$ this is the (nonmanifest) infinitesimal time translation considered in the previous example; the associated charge is minus the hamiltonian.
- For all other cases the symmetry is nonmanifest in the colloquial sense as well. Physically, the additional conserved charges are clear. Choose a basis which diagonalizes the metric. Then we have m identical oscillators moving independently. Each has a conserved energy, and these energies are the charges associated to special choices of A and B. Other charges are generated from these and the angular momenta by Poisson bracket.
- We have specified a finite dimensional vector space of vector fields on $\mathcal{F}$. It is not closed under Lie bracket, but rather generates an infinite dimensional Lie algebra of vector fields. However, if we look on-shell—that is, on the space of classical solutions $\mathcal{M}$—then this vector space of vector fields *is* closed under Lie bracket. In fact, the resulting Lie algebra is isomorphic to the Lie algebra of the unitary group $U(k)$.

The reader may wish to return to this example after learning more about supersymmetry. For some supersymmetric theories there is a formulation in which the supersymmetry is manifest (superspacetime formulation), and then another formulation (component formulation) where the supersymmetry is not manifest but still the supersymmetry transformations "close off-shell". (In the harmonic oscillator example the vector fields with $B = 0$ close off-shell; they form a Lie algebra isomorphic to the Lie algebra of $O(k)$.) The superparticle we consider next is an example of such a supersymmetric theory; we will discuss the superspacetime formulation in the next section. For other supersymmetric theories, however, there is

no known formulation which makes the supersymmetry manifest and the supersymmetry transformations only close on-shell. (In the harmonic oscillator example the entire vector space of symmetries only closes on-shell.) In these examples brackets of the infinitesimal supersymmetry transformations off-shell produce an infinite dimensional Lie algebra, as for the harmonic oscillator.

I do not know if there is an alternative formulation of this system of harmonic oscillators which makes the $U(k)$ symmetry manifest. By "alternative formulation" we mean the following. There are often different lagrangians with different spaces of fields which describe the same classical field theory—the same space of classical solutions, symplectic structure, Noether currents, etc. For example, instead of (2.27) we can consider a theory with two fields $x, y\colon M^1 \to X$ and lagrangian

$$L = \left\{ \langle y, \dot{x} \rangle - \frac{1}{2}\left(|x|^2 + |y|^2\right) \right\} dt.$$

Notice that y only enters algebraically—no derivatives—and it can be eliminated using its equation of motion $y = \dot{x}$ to recover (2.27). This is simply the lagrangian version of the device which converts a second order differential equation into a system of first order equations (which is what happens to the equations of motion for this system). But in this formulation still only the $O(k)$ symmetry is manifest.

Example 4 (a supersymmetry at last!). For convenience we collate the formulas for the superparticle:

$$\begin{aligned}
L_2 &= \left\{ \frac{1}{2}|\dot{x}|^2 + \frac{1}{2}\langle \psi, \nabla_{\dot{x}}\psi \rangle \right\} dt \\
\gamma_2 &= \langle \dot{x}, \delta x \rangle + \frac{1}{2}\langle \psi, \delta_\nabla \psi \rangle \\
\omega_2 &= \langle \delta_\nabla \dot{x} \wedge \delta x \rangle + \frac{1}{2}\langle \delta_\nabla \psi \wedge \delta_\nabla \psi \rangle + \frac{1}{4}\langle \psi, R(\delta x \wedge \delta x)\psi \rangle.
\end{aligned}$$

We also recall

$$\delta L_2 = -d\gamma_2 + \langle \delta x, -\nabla_{\dot{x}}\dot{x} + \frac{1}{2}R(\psi,\psi)\dot{x} \rangle \wedge dt + \langle \delta_\nabla \psi, \nabla_{\dot{x}}\psi \rangle \wedge dt.$$

As for the classical particle, infinitesimal time translation is both a manifest symmetry ξ and a nonmanifest symmetry $\hat{\xi}$. The nonmanifest symmetry is the vector field

$$\begin{aligned}
\iota(\hat{\xi})\delta x &= -\dot{x} \\
\iota(\hat{\xi})\delta \psi &= -\nabla_{\dot{x}}\psi.
\end{aligned}$$

The direct computation that $\hat{\xi}$ is a nonmanifest infinitesimal symmetry is a good exercise in calculus with odd variables. The associated Noether charge is

$$j_{\hat{\xi}} = -\frac{1}{2}|\dot{x}|^2$$

and so the hamiltonian is

$$H = \frac{1}{2}|\dot{x}|^2.$$

Notice that the fermion does not enter into the formula for the hamiltonian. Quite generally, *kinetic energy terms in hamiltonians do not involve fermions*, whereas fermions may enter into potential energy terms.

Now we introduce an odd vector field $\hat{Q}$ which is a nonmanifest symmetry. It is our first example of a classical supersymmetry. By introducing an auxiliary odd parameter η we can consider instead the even vector field $\hat{\zeta} = \eta\hat{Q}$. The definition of $\hat{\zeta}$ is

$$\begin{aligned} \iota(\hat{\zeta})dt &= 0 \\ \iota(\hat{\zeta})\delta x &= -\eta\psi \\ \iota(\hat{\zeta})\delta_\nabla\psi &= +\eta\dot{x}. \end{aligned} \tag{2.28}$$

There is no component in the M^1 direction; $\hat{\zeta}$ projects to a vector field on $\mathcal{F}$. We leave the reader to compute that

$$[\hat{Q}, \hat{Q}] = 2\hat{\xi}. \tag{2.29}$$

Thus the two-dimensional vector space generated by the vector fields $\hat{Q}, \hat{\xi}$ closes off-shell. It is the simplest example of a supersymmetry algebra.

Now we do part of the computation that $\hat{\zeta}$ is a nonmanifest infinitesimal symmetry.

$$\begin{aligned} \mathrm{Lie}(\hat{\zeta})L_2 &= \iota(\hat{\zeta})\delta L_2 \\ &= d\iota(\hat{\zeta})\gamma_2 + \langle\eta\psi, \nabla_{\dot{x}}\dot{x}\rangle\, dt - \frac{1}{2}\langle\psi, R(\psi,\psi)\dot{x}\rangle\, dt + \langle\eta\dot{x}, \nabla_{\dot{x}}\psi\rangle\, dt \\ &= d\iota(\hat{\zeta})\gamma_2 + \eta\big(\langle\psi, \nabla_{\dot{x}}\dot{x}\rangle + \langle\nabla_{\dot{x}}\psi, \dot{x}\rangle\big)\, dt \\ &= d\big(\iota(\hat{\zeta})\gamma_2 + \eta\langle\psi, \dot{x}\rangle\big). \end{aligned}$$

In the third step we used the Bianchi identity $R(\psi,\psi)\psi = 0$. The conserved Noether charge is

$$\begin{aligned} j_{\hat{\zeta}} &= \iota(\hat{\zeta})\gamma_2 - \big(\iota(\hat{\zeta})\gamma_2 + \eta\langle\psi, \dot{x}\rangle\big) \\ &= -\eta\langle\psi, \dot{x}\rangle. \end{aligned}$$

Now just as the hamiltonian H is *minus* the charge associated to $\hat{\xi}$, we define the *supercharge* Q to be *minus* the current associated to $\hat{Q}$ (recall $\hat{\zeta} = \eta\hat{Q}$):

$$Q = -j_{\hat{Q}} = \langle\psi, \dot{x}\rangle.$$

We leave the reader to check the Poisson bracket

$$\{Q, Q\} = -2H. \tag{2.30}$$

The supercharge Q does not have an immediate physical interpretation in terms of energy, momentum, or the like. We might not expect that here since our model is not physical, but even in more physical field theory models there is no immediate physical significance to the classical supercharge simply because there are in some physical sense no classical fermions. There are important physical consequences of the fermions on the bosons, however. For example, the energy H is nonnegative since it is a square. (The sign conventions work out so that (2.30) does give positive—rather than negative—energy!) We will see a more direct *geometric* meaning for the supercharge when we quantize.

Superspacetime and manifest supersymmetry

Equation (2.29) can be interpreted loosely as saying that $\hat{Q}$ is a square root of $\hat{\xi}$. Since $\hat{\xi}$ is the effect on fields of infinitesimal time translation, we are led to wonder if there is a square root of infinitesimal time translation whose action on fields induces $\hat{Q}$. Of course, this cannot happen on the real line, nor on any ordinary manifold. The "square" of a vector field—one-half its bracket with itself—vanishes for even vector fields, since the Lie bracket is skew-symmetric. So we replace M^1 by a supermanifold and seek an odd vector field as the square root. Then we write the fields in our theory as functions on that supermanifold so that such an odd vector field induces an action on fields.

Such a construction exists and it leads to a formulation of the superparticle in which the supersymmetry is manifest. The supermanifold we seek goes under the name *superspacetime*. (Of course, *superspacetime* is usually simply *superspace*. In this example, *supertime* would be even more appropriate.) The superspacetime formulation of supersymmetric theories goes back to the mid '70s when it was introduced by Salam and Strathdee. Be warned that not all supersymmetric theories have a superspacetime formulation, as mentioned earlier. We give a general construction of superspacetime in the next lecture. Here we simply introduce the superspacetime relevant to the superparticle.

Let $M^{1|1}$ denote the affine space whose ring of functions is

$$C^\infty(M^{1|1}) = C^\infty(M^1)[\theta]$$

for an odd variable θ. Note:

- θ is *not* an *auxiliary* odd variable—it is a bona fide odd function on $M^{1|1}$.
- In our *component* formulation of the superparticle we had fields x, ψ which map an ordinary manifold into a supermanifold. Now we consider a *superfield* formulation in which the field Φ maps a supermanifold into an ordinary manifold. (See Figure 6.)

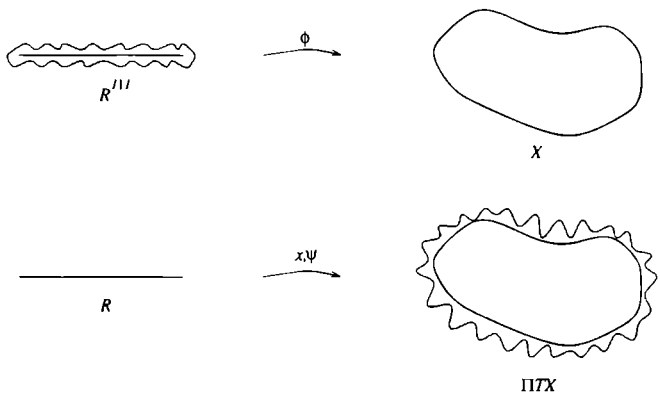

Figure 6: The superfield Φ and the component fields x, ψ

Let t be a standard affine coordinate on M^1, so that t, θ are global coordinates on $M^{1|1}$. Let

$$i\colon M^1 \hookrightarrow M^{1|1}$$

be the inclusion defined by

$$\begin{aligned} i^*t &= t \\ i^*\theta &= 0. \end{aligned}$$

We introduce a global framing of $M^{1|1}$ by the vector fields

$$\begin{aligned} \partial_t &= \frac{\partial}{\partial t} \\ D &= \partial_\theta - \theta\partial_t = \frac{\partial}{\partial\theta} - \theta\partial_t. \end{aligned}$$

Here ∂_t is even and D is odd. It is unfortunate that the vector field 'D' has the same symbol as the total differential considered in earlier parts of the lecture. *Tant pis*, there just aren't enough dees in the world! We also introduce the odd vector field

$$\tau_Q = \partial_\theta + \theta\partial_t.$$

We will see in the next lecture that $M^{1|1}$ is the supermanifold underlying a Lie group on which $\{\partial_t, D\}$ is a basis of left invariant vector fields and $\{\partial_t, \tau_Q\}$ a basis of right invariant vector fields. They satisfy the bracket relations

$$\begin{aligned} [D, D] &= -2\partial_t \\ [\tau_Q, \tau_Q] &= +2\partial_t \\ [D, \tau_Q] &= 0. \end{aligned}$$

The vector field ∂_t commutes with both D and τ_Q.

We now formulate a field theory with spacetime $M^{1|1}$ and space of fields

$$\mathcal{F}_2' = \{\Phi\colon M^{1|1} \longrightarrow X\},$$

where X is our fixed Riemannian manifold. In Lecture 1 we discussed what such a map looks like in case we replace $M^{1|1}$ by $M^{0|1}$, a one-dimensional odd vector space. From that discussion we see that Φ is intuitively a path in X and an odd tangent vector field along the path. In other words, it is exactly the same data encoded by the fields x, ψ in our previous formulation. We recover these *component fields* from Φ by restricting D-derivatives of the superfield to the underlying Minkowski spacetime:

$$\begin{aligned} x &:= i^*\Phi \\ \psi &:= i^*D\Phi. \end{aligned} \tag{2.31}$$

The lagrangian density in superspacetime is

$$\mathcal{L} = |dt|\, d\theta \left\{-\frac{1}{2}\langle D\Phi, \partial_t\Phi\rangle\right\} = |dt|\, d\theta\, \ell.$$

Here $|dt|\, d\theta$ is a bi-invariant density on $M^{1|1}$—it is invariant under ∂_t, D, τ_Q. We will not give a detailed development of densities and integration on supermanifolds

in these lectures. We finesse the issue below. Also, as above we orient the line and replace $|dt|$ with dt.

Before integrating we pause to point out that we have achieved our goal of making the supersymmetry manifest. Namely, the lagrangian $\mathcal{L}$ is invariant under the vector fields $\{\zeta, \xi\}$ on $\mathcal{F}_2' \times M^{1|1}$ induced by $\{\eta\tau_Q, \partial_t\}$. Explicitly, we have

$$\begin{aligned} \iota(\zeta)dt &= \eta\theta & \iota(\xi)dt &= 1 \\ \iota(\zeta)d\theta &= \eta & \iota(\xi)d\theta &= 0 \\ \iota(\zeta)\delta\Phi &= -\eta\tau_Q\Phi & \iota(\xi)\delta\Phi &= -\partial_t\Phi. \end{aligned}$$

The invariance of $\mathcal{L}$ follows *a priori* from the fact that τ_Q, ∂_t commute with D, ∂_t and the fact that the density $|dt|\, d\theta$ is invariant.

Now to the integration. We define the *component lagrangian* L from the superspacetime lagrangian $\mathcal{L}$ by "integrating out" the odd variable θ. This is the *Berezin integral* which in this case amounts to

$$L = (i^* D\ell)\, dt.$$

(This is our finesse: We simply introduce this definition without more explanation. We will say a bit more in Lecture 4.) So we compute

$$\begin{aligned} i^* D\ell &= -\frac{1}{2} i^* D\langle D\Phi, \partial_t\Phi\rangle \\ &= -\frac{1}{2} i^* \Big\{ \langle \nabla_D D\Phi, \partial_t\Phi\rangle - \langle D\Phi, \nabla_D \partial_t\Phi\rangle \Big\} \\ &= -\frac{1}{2} i^* \Big\{ -|\partial_t\Phi|^2 - \langle D\Phi, \nabla_{\partial_t} D\Phi\rangle \Big\} \\ &= \frac{1}{2}|\dot{x}|^2 + \frac{1}{2}\langle \psi, \nabla_{\dot{x}}\psi\rangle. \end{aligned}$$

Hence we recover the superparticle component lagrangian L_2 which we introduced in (2.9).

The reader would do well to compute that other features of the superspacetime model match the component formulation. For example, compute the action of τ_Q on the component fields using the definition (2.31) of the component fields to recover (2.28). This computation is a bit tricky, but can be viewed as a problem in ordinary differential geometry—the odd variables cause no additional difficulties. Also, you can do the classical mechanics directly on $M^{1|1}$: Compute γ, ω, the equations of motion, the supercharge, etc. This is a nontrivial exercise in calculus on supermanifolds; the component formulas in the text may be used as a check.

Quantization of the superparticle

We make only brief heuristic remarks about the quantization. After all, we promised in Lecture 1 that we would only quantize symplectic affine spaces and here we are with a curved example! We work in the component formulation. Quite generally, it is easier to see the physics—in particular, the quantization—in components, whereas the formal properties due to supersymmetry are manifest in the superspacetime formulation.

We remarked in (2.15) that if we fix a time then we can identify the supermanifold of classical solutions with the symplectic supermanifold

$$\pi^*\Pi TX \longrightarrow TX; \tag{2.32}$$

the symplectic form is (2.16). For $X = \mathbb{R}^m$ this combines the two examples we studied in Lecture 1. We quantize in stages, first quantizing the fibers of (2.32) and then the base. Now the fiber over a fixed element of T_xX is the odd vector space ΠT_xX with the symplectic form induced from the metric. We found previously that if $\dim X$ is even its quantization is the spin space $S_x = S_x^0 \oplus S_x^1$. In fact, only the projective space is well-defined. Thus we obtain the projective spin bundle of X, pulled back over TX. To go further we need to assume that this can be lifted to an honest graded vector bundle, i.e., we need to assume that the manifold X has a spin structure. The quantization of the base TX we take to be the space of L^2 functions on X. Therefore, the quantization of the fibered space is the "twisted tensor product" of the L^2 functions on X and the spinor bundle on X, i.e., it is the Hilbert space

$$\mathcal{H} = \mathcal{H}^0 \oplus \mathcal{H}^1$$

of L^2 spinor fields on X.

Now we ask for the self-adjoint operators $\hat{Q}, \hat{H}$ which quantize the supercharge Q and the hamiltonian H. Roughly speaking, the quantization $\hat{\psi}$ of ψ is Clifford multiplication and the quantization $\hat{\dot{x}}$ of $\dot{x}$ is the covariant derivative. Hence the quantization of $Q = \langle \psi, \dot{x} \rangle$ is the Dirac operator $\hat{Q}$ and the quantization of $H = \frac{1}{2}|\dot{x}|^2$ is the Dirac laplacian. Notice that (2.30) at the classical level is reflected by the operator equation which asserts that the Dirac laplacian is the square of the Dirac operator. There is much more to say here, but these brief remarks will have to suffice.

As promised, the supercharge has more immediate geometric meaning in the quantum story ($\hat{Q}$) than in the classical story (Q). In fact, this quantum mechanical model was used in the early '80s to give a physicists' proof of the Atiyah-Singer index theorem for Dirac operators.

Yet another word about signs

The definition of Lie bracket on a manifold is in terms of the action of vector fields on functions rather than points. This leads to a sign when we consider *left* group actions. Namely, if G is a Lie group which acts on the left on a manifold N, then the induced map $\mathfrak{g} \to \mathcal{X}(N)$ from the Lie algebra of G to the Lie algebra of vector fields on N is an *anti*homomorphism: it reverses the sign of brackets. If N is symplectic and we lift this antihomomorphism to a moment map $\mathfrak{g} \to \Omega^0(N)$, then the moment map is also an *anti*homomorphism.

We follow standard mathematical usage which has symmetries acting on the left, and so in a lagrangian field theory if we have a group G of symmetries we take it to act on the left on $\mathcal{F} \times M$. This explains the choice of sign in (2.26), for example. This choice—between left and right actions—is often referred to as the choice between the passive point of view and the active point of view. Now in classical lagrangian field theory we pass from an infinitesimal symmetry λ to a vector field ξ_λ on $\mathcal{F} \times M$ to a Noether current j_λ to a Noether charge Q_λ. When we quantize we then pass from the Noether charge Q_λ to a self-adjoint operator $\hat{Q}_\lambda$ on a

Hilbert space $\mathcal{H}$, but since they do not form a Lie algebra we pass to a skew-adjoint operator $-i\hat{Q}_\lambda$. We indicate these maps in the following sequence:

$$\begin{array}{ccccccccccc}
\mathfrak{g} & \longrightarrow & \mathcal{X}(\mathcal{F}\times M) & \longrightarrow & \Omega^{-1}(\mathcal{F}\times M) & \longrightarrow & \Omega^0(\mathcal{M}) & \longrightarrow & \text{Self-Adj}(\mathcal{H}) & \longrightarrow & \text{Skew-Adj}(\mathcal{H}) \\
\lambda & \longmapsto & \xi_\lambda & \longmapsto & j_\lambda & \longmapsto & Q_\lambda & \longmapsto & \hat{Q}_\lambda & \longmapsto & -i\hat{Q}_\lambda \\
[\lambda_1,\lambda_2] & \longmapsto & -[\xi_{\lambda_1},\xi_{\lambda_2}] & \longmapsto & -\{j_{\lambda_1},j_{\lambda_2}\} & \longmapsto & -\{Q_{\lambda_1},Q_{\lambda_2}\} & & \longmapsto & & [-i\hat{Q}_{\lambda_1},-i\hat{Q}_{\lambda_2}]
\end{array}$$

The bottom line tells which maps are homomorphisms and which are antihomomorphisms. In particular, the map from infinitesimal symmetries to skew-adjoint operators in the quantum theory is a homomorphism. Note that we omitted the factor of $\hbar$ and the correction term which appear in (1.18).

These signs follow from the fact that we chose a left group action and from our choice of sign in (1.18). We remind the reader of a convention regarding the hamiltonian: it is *minus* the conserved current associated to time translation. The same holds for the supercharges in a supersymmetric theory and for the entire *energy-momentum tensor* and *supercurrent* in a field theory.

LECTURE 3
Supersymmetry in Various Dimensions

In the first two lectures we discussed some general features of field theory, but our examples were all from mechanics. The underlying "spacetime" of the theories considered is just time; the "space" in those models is the manifold in which paths take values. In the remaining lectures we turn to field theory proper—spacetimes with positive space dimension. Put differently, the spacetime dimension n is at least 2. In this lecture we present additional general material needed to treat fields in supersymmetric theory; the remaining lectures will consider specific supersymmetric models.

We mathematicians are fond of abstract structures which apply in many different examples. While (supersymmetric) field theory exhibits such general features—and we try to bring them out as much as possible—the whole subject is centered around particular examples which to a large extent resist abstraction. In particular, many properties depend strongly on the spacetime dimension n. For this reason some readers may find that parts of the remaining lectures resemble lessons in geography or taxonomy. We encounter this immediately in our discussion of spinors in Minkowski spacetime. It is necessary background to introduce fermion fields in Minkowski spacetime, and the particularities of spinors in various dimensions influence the models which exist. They also enter into the general construction of *superspacetime* and the *super Poincaré group*, which is our second topic. The Hilbert space of a supersymmetric *quantum* field theory carries a representation of the super Poincaré group, so we give a general discussion and some examples of such representations. Our discussion of representations of the ordinary Poincaré group in Lecture 1 serves as a model here.

In the second half of the lecture we turn from quantum particles to *classical* fields. We give some general discussion and survey the different supersymmetric field theories. Then we turn to (nonsupersymmetric) *free* field theories and their quantization. The space of classical solutions to a free theory is a symplectic vector or affine space. We then see how to get the one-particle quantum Hilbert space and the associated Fock space for these free theories. Here the corresponding discussion for free particles in Lecture 1 serves as a model. But in that discussion the symplectic vector spaces we quantized are finite dimensional; here in field theory they are infinite dimensional.

Spinors on Minkowski spacetime

n	Spin $(1, n-1)$	minimal real spin representation S	$\dim S$	V in terms of S^*
1	$\mathbb{Z}/2$	S real	1	$V \cong (S^*)^{\otimes 2}$
2	$\mathbb{R}^{>0} \times \mathbb{Z}/2$	S^+, S^- real	1	$V \cong (S^{+*})^{\otimes 2} \oplus (S^{-*})^{\otimes 2}$
3	$SL(2, \mathbb{R})$	S real	2	$V \cong \mathrm{Sym}^2 S^*$
4	$SL(2, \mathbb{C})$	$S_{\mathbb{C}} \cong S' \oplus S''$	4	$V_{\mathbb{C}} \cong S'^* \otimes S''^*$
5	$Sp(1,1)$	$S_{\mathbb{C}} \cong S_0 \otimes_{\mathbb{C}} W$	8	$\bigwedge^2 S_0^* \cong \mathbb{C} \oplus V_{\mathbb{C}}$
6	$SL(2, \mathbb{H})$	$S_{\mathbb{C}}^{\pm} \cong S_0^{\pm} \otimes_{\mathbb{C}} W$	8	$V_{\mathbb{C}} \cong \bigwedge^2 S_0^{+*} \cong (\bigwedge^2 S_0{}^*)^*$
7		$S_{\mathbb{C}} \cong S_0 \otimes_{\mathbb{C}} W$	16	$\bigwedge^2 S_0^* \cong V_{\mathbb{C}} \oplus \bigwedge^2 V_{\mathbb{C}}$
8		$S_{\mathbb{C}} \cong S' \oplus S''$	16	$S'^* \otimes S''^* \cong V_{\mathbb{C}} \oplus \bigwedge^3 V_{\mathbb{C}}$
9		S real	16	$\mathrm{Sym}^2 S^* \cong \mathbb{R} \oplus V \oplus \bigwedge^4 V$
10		S^+, S^- real	16	$\mathrm{Sym}^2 (S^{\pm})^* \cong V \oplus \bigwedge^5_{\pm} V$
11		S real	32	$\mathrm{Sym}^2 S^* \cong V \oplus \bigwedge^2 V \oplus \bigwedge^5 V$

Table 1: Spinors in Lorentz signature

We need to know a bit about the *real* spin representations of the Lorentz group $\mathrm{Spin}(1, n-1)$. This enters both the discussion of spinor fields and also the construction of the super Poincaré group. The basic properties are summarized in Table 1. In this table V is the vector representation of $\mathrm{Spin}(1, n-1)$, a real n-dimensional vector space with an inner product of signature $(1, n-1)$. We highlight some of the important information:

- The first column is the dimension n of spacetime. The reason we stop at $n = 11$ will emerge later. In Lecture 4 we focus completely on theories in dimension 3 or less. In Lecture 5 we consider theories in 4 dimensions in detail, and theories in higher dimension in somewhat less detail. Therefore, the reader is well-advised to study $n \leq 4$ carefully.
- Many of us are more familiar with properties of spinors for the compact spin group $\mathrm{Spin}(n)$. There is a mod 8 periodicity—a form of Bott periodicity — and for the Lorentz group here we also find a mod 8 periodicity. For example, the column labeled '$\dim S$' has the property that $\dim S$ for spacetime dimension $n+8$ is 16 times $\dim S$ for spacetime dimension n. Some properties—for example, the reality conditions—are shifted from the compact case.
- The second column reflects the special isomorphisms among low rank simple Lie groups—they identify the orthogonal groups in dimension at most 6 in terms of other groups. Again, the corresponding list of compact groups is more familiar, as are the exceptional isomorphisms of the complex simple groups. Here we list the exceptional isomorphism for the Lorentz groups, which are noncompact real forms. Notice the neat appearance of $SL(2)$ over the reals, complexes, and quaternions in dimensions $n = 3, 4, 6$. The minimal real spin representations have real dimensions $2, 4, 8$. We will see

that dimensions $3, 4, 6$ are the critical dimensions for theories with $2, 4, 8$ supersymmetries. (The maximal dimension in which there is a real spinor representation of dimension 16 is $n = 10$. There is a case to be made that $\mathrm{Spin}(1,9)$ should be viewed as $SL(2)$ over the octonions. Notice that the sequence of dimensions $n = 3, 4, 6, 10$ is closely related to the dimensions $1, 2, 4, 8$ of the division algebras, which reflects the fact that the Clifford algebra for signature $(1, n-1)$ is closely related to the Clifford algebra for signature $(0, n-2)$.)

- The third column lists the smallest real spin representation. If for a moment we consider *complex* spin representations, then there is a mod 2 (Bott) periodicity: In odd dimensions there is a unique irreducible complex spin representation and in even dimensions there are two such irreducible half spin representations. In each dimension we ask if the complex spin representation carries a real or quaternionic structure. The answer depends on $n \pmod 8$ and is indicated in the chart. For $n \equiv 1, 2, 3 \pmod 8$ the representation is real, for $n \equiv 4, 8 \pmod 8$ it is complex, and for $n \equiv 5, 6, 7 \pmod 8$ it is quaternionic. (We call the representation complex if it carries no real or quaternionic structure.)
- We write the complexification of a quaternionic spin representation S as $S_{\mathbb{C}} \cong S_0 \otimes_{\mathbb{C}} W$, where W is a fixed two-dimensional complex vector space on which the quaternions $\mathbb{H}$ act (on the right). Also, W carries a fixed complex symplectic form. The quaternionic dimension of S is half the complex dimension of S_0. Thus in $n = 6$ dimensions the spin representation is two-dimensional quaternionic. Note that we always use 'S' to denote a *real* spin representation, which for $n = 6$ is the underlying 8-dimensional real representation of the 2-dimensional quaternionic representation.
- S is self-dual—$S \cong S^*$—except in dimensions $n \equiv 2, 6 \pmod 8$ where $(S^+)^* \cong S^-$.
- In even dimensions n there are two *complex* half spin representations S^+, S^-. For $n \equiv 2 \pmod 8$ each of them is real and the underlying real representations are inequivalent. For $n \equiv 6 \pmod 8$ each is quaternionic and the underlying real representations are inequivalent. For $n \equiv 4, 8 \pmod 8$ they are complex conjugate (here denoted S' and S'') and the underlying real representations are isomorphic. The half spin representations are often called *chiral*.
- Physicists use the term *Dirac spinors* for elements in the full complex spin representation, *Weyl spinors* for elements in a half spin representation for n even, and *Majorana spinors* for elements in a real spin representation. For example, they say that *Majorana-Weyl spinors* exist only for $n \equiv 2 \pmod 8$.
- The last column of the table expresses the vector space V in terms of the square of dual spinors. One feature on Minkowski spacetime is that in every dimension there is a *symmetric* equivariant pairing

$$\Gamma : S^* \otimes S^* \longrightarrow V. \tag{3.1}$$

There is no other signature for which such a pairing exists for all real spin representations. This pairing enters into our discussion of superspacetime and the super Poincaré group.

- There also exists a symmetric pairing

$$\tilde{\Gamma}\colon S \otimes S \longrightarrow V \tag{3.2}$$

on spinors. We use it to construct the Dirac form which occurs in the kinetic term of the classical lagrangian for spinor fields. The two pairings (3.1) and (3.2) satisfy a Clifford-type relation which is most easily expressed in indices. With respect to bases $\{e_\mu\}$ of V and $\{f^a\}$ of S, we write

$$\Gamma(f_a, f_b) = \Gamma^\mu_{ab} e_\mu, \qquad \tilde{\Gamma}(f^a f^b) = \tilde{\Gamma}^{\mu ab} e_\mu.$$

Here $\{f^a\}$ is the dual basis of S^*. Then the Clifford relation is

$$\Gamma^\mu_{ab}\tilde{\Gamma}^{\nu bc} + \Gamma^\nu_{ab}\tilde{\Gamma}^{\mu bc} = 2g^{\mu\nu}\delta^c_a, \tag{3.3}$$

where $g^{\mu\nu}$ is the inverse metric. For field theory the symmetric pairings (3.1) and (3.2) are more fundamental than Clifford multiplication.

- As mentioned earlier, in dimensions $n \not\equiv 2, 6 \pmod 8$ there is a duality pairing

$$\epsilon\colon S \otimes S \longrightarrow \mathbb{R} \tag{3.4}$$

for any minimal real spin representation S. The pairing is skew-symmetric for $n \equiv 3, 4 \pmod 8$ and symmetric for $n \equiv 1, 5, 7, 8 \pmod 8$. In these cases we normalize (3.4) so that the pairings Γ (3.1) and $\tilde{\Gamma}$ (3.2) are related by

$$\tilde{\Gamma}^{\mu ab} = \Gamma^\mu_{a'b'}\epsilon^{aa'}\epsilon^{bb'}. \tag{3.5}$$

In dimensions $n \equiv 2, 6 \pmod 8$ there is a nondegenerate pairing of S^+ with S^-; then Γ for $S^\pm$ and $\tilde{\Gamma}$ for $S^\mp$ are related as in (3.5).

- Dimension $n = 2$ is special in that the vector representation V is reducible: $V \cong V^+ \oplus V^-$. This does not happen in any other dimension, and it is the starting point for many special features of two-dimensional field theory.
- What is listed in the table is the *minimal* real spin representation. In many situations the real spin representation used is a direct sum of several copies of minimal representations.

Superspacetime and the super Poincaré group

Minkowski spacetime $\check{M}^n$ is the affine space underlying the translation group V. (The "check" notation serves to distinguish ordinary Minkowski spacetime from the super version introduced shortly.) The Poincaré group acts as a symmetry group of $\check{M}^n$. In this section we construct super analogs. We introduce a super Lie group of "translations" which is noncommutative. The underlying supermanifold is super Minkowski spacetime, or *superspacetime* for short. Then we define the super Poincaré group as a certain group of symmetries of super Minkowski spacetime.

To define superspacetime we fix the following data:

(3.6)

V	vector space with Lorentz metric
$C \subset V$	positive cone of timelike vectors
S	real spin representation of $\mathrm{Spin}(V) \cong \mathrm{Spin}(1, n-1)$
$\Gamma\colon S^* \otimes S^* \longrightarrow V$	symmetric equivariant pairing

We require that the pairing satisfy the positivity condition

$$\begin{aligned} &\Gamma(s^*, s^*) \in \overline{C}, && s^* \in S^*, \\ &\Gamma(s^*, s^*) = 0 && \text{if and only if } s^* = 0. \end{aligned} \tag{3.7}$$

Notice that the choice of C among the two halves of the lightcone is simply a choice of a positive arrow of time. In terms of bases the hypothesis is that

$$\Gamma^0_{aa} > 0 \qquad \text{for all } a,$$

where $e_0 \in V$ points in the direction of positive time. From the Clifford relation (3.3) we deduce

$$\tilde{\Gamma}^{0aa} > 0 \qquad \text{for all } a,$$

and in fact $\tilde{\Gamma}$ satisfies (3.7).

The spin representation S need *not* be the minimal ones listed in Table 1. It may be a sum of several copies of the minimal one(s).

We remark that only the conformal structure underlying the Lorentz metric on Minkowski spacetime enters into this data.

Now define the graded Lie algebra

$$\mathcal{L} = V \oplus S^* \tag{3.8}$$

to be the graded vector space shown with V central and nontrivial bracket

$$[f_1, f_2] = -2\Gamma(f_1, f_2), \qquad f_1, f_2 \in S^*. \tag{3.9}$$

Let $\exp \mathcal{L}$ be the corresponding super Lie group and M the underlying supermanifold. The splitting (3.8) means that as a manifold

$$M \cong \check{M} \times \Pi S^*, \tag{3.10}$$

where $\check{M}$ is the affine space underlying V.

We make some remarks about this construction of superspacetime M:

- A super Lie group is best understood with auxiliary odd parameters: for each commutative superalgebra R the "R-points"—that is, the group whose Lie algebra is the ordinary Lie algebra $(R \otimes \mathcal{L})^0$—form an ordinary group. It is a good exercise to write the group law in this group using the Hausdorff formula. Since $\mathcal{L}$ is only mildly nonabelian, the formulas are quite simple.
- The positivity condition (3.7) has an important consequence: Minus time translation can be written as a sum of squares of odd elements in the Lie algebra $\mathcal{L}$. (The square of an odd element f is $\frac{1}{2}[f, f]$.) When we quantize a supersymmetric theory, the self-adjoint quantum operator which corresponds

to minus time translation is the hamiltonian $\hat{H}$. From this Lie algebra statement we conclude that $-i\hat{H}$ is the sum of squares of odd skew-adjoint operators, and with our sign conventions this means that $\hat{H}$ is nonnegative. (See the end of Lecture 2 and (1.14).) So *the energy (hamiltonian) in a supersymmetric field theory is nonnegative*. This also holds in the classical theory, as we will see later (at least in examples).

- The sign conventions in the physics literature vary, certainly, but there is probably unanimity in using a positive sign in (3.9) as opposed to our negative sign! *Caveat emptor!* Beware when comparing our signs to those in any physics reference.
- Consider the left-invariant odd distribution $\mathcal{S}^*$ on M which at the origin is $S^* \subset \mathcal{L}$. We might call it a "conformal structure" on M. The group of diffeomorphisms of M which preserve $\mathcal{S}^*$ is the *superconformal group*; it contains left translations and extensions to superspacetime of conformal transformations on (open subsets of) Minkowski spacetime.

Introduce bases $\{f_a\}$ for S^* and $\{e_\mu\}$ for V. We write the symmetric pairing Γ as before:

$$\Gamma(f_a, f_b) = \Gamma^\mu_{ab} e_\mu.$$

In the physics literature it is customary to denote e_μ, f_a by P_μ, Q_a, and we sometimes follow that convention. Thus we write (3.9) as

$$[Q_a, Q_b] = -2\Gamma^\mu_{ab} P_\mu.$$

Let x^μ be the linear coordinates on V and θ^a the odd linear coordinates on ΠS^* corresponding to the given bases. Then $\{x^\mu, \theta^a\}$ is a global coordinate system on superspacetime M. Let ∂_μ and $\partial/\partial\theta^a$ be the corresponding coordinate vector fields. We will usually express our computations in terms of a global framing of M rather than in terms of these global coordinates. We use the basis $\{\partial_\mu, D_a\}$ of *left* invariant vector fields to write lagrangians, and the commuting basis $\{\partial_\mu, \tau_{Q_a}\}$ of *right* invariant vector fields to write symmetries. Note that ∂_μ is both left and right invariant since V is central. The vector fields D_a and τ_{Q_a} are given by the formulas

$$\begin{aligned} D_a &= \frac{\partial}{\partial\theta^a} - \Gamma^\mu_{ab}\theta^b\partial_\mu \\ \tau_{Q_a} &= \frac{\partial}{\partial\theta^a} + \Gamma^\mu_{ab}\theta^b\partial_\mu, \end{aligned}$$

as the reader may check. The nontrivial brackets are

$$\begin{aligned} [D_a, D_b] &= -2\Gamma^\mu_{ab}\partial_\mu \\ [\tau_{Q_a}, \tau_{Q_b}] &= +2\Gamma^\mu_{ab}\partial_\mu. \end{aligned}$$

The brackets of the left invariant D_a are as in the Lie algebra $\mathcal{L}$; brackets of the right invariant τ_{Q_a} are opposite. Also

$$[D_a, \tau_{Q_b}] = 0$$

since left invariant vector fields commute with right invariant vector fields.

The coordinates and vector fields are all *real*. In certain cases, such as the $n = 4$ superspacetime we discuss in Lecture 5, it is convenient to use a global framing by *complex* vector fields.

For $n = 1$ and the one-dimensional spin representation S we recover the superspacetime and superalgebra encountered in Lecture 2 when we discussed the superparticle. The even part of the algebra is generated by infinitesimal time translation ∂_t and the odd part by an element Q with

$$[Q, Q] = -2\partial_t.$$

The *super Poincaré algebra* is the graded Lie algebra

$$\mathfrak{p} = \big(V \oplus \mathfrak{so}(V)\big) \oplus S^*. \tag{3.11}$$

It contains both $\mathcal{L}$ and the ordinary Poincaré algebra as subalgebras. The nontrivial brackets are the representation of $\mathfrak{so}(V)$ on V and on S^* as well as the bracket (3.9) on odd elements. The *super Poincaré group* P is the semi-direct product $\mathrm{Spin}(V) \ltimes \exp(\mathcal{L})$.

A word about nomenclature: Let $s = \dim S$. Then we usually use '$M^{n|s}$' to denote superspacetime and '$P^{n|s}$' to denote the corresponding super Poincaré group. This fixes V and S up to isomorphism in cases where there is only one minimal real spin representations. But in cases where there are chiral spin representations—for $n \equiv 2, 6 \pmod 8$—we use the notations '$M^{n|(s^+,s^-)}$' and '$P^{n|(s^+,s^-)}$' where the decomposition $s = s^+ + s^-$ indicates the dimension of the spaces of chiral supersymmetries. For example, the supersymmetry group relevant to the superparticle is $P^{1|1}$. As other examples of minimal supersymmetry groups we cite $P^{3|2}$, $P^{4|4}$, and $P^{6|(8,0)}$.

Our notation follows the modern convention of labeling supersymmetric theories by the dimension of the space of supersymmetries—the number of basis elements Q_a. There is another system which refers to "$N = 1$" supersymmetry, "$N = 2$" supersymmetry, etc. Here the nomenclature is dimension dependent. '$N = 1$' usually refers to the minimal amount of supersymmetry, '$N = 2$' to twice as much, and so on. Thus $N = 1$ supersymmetry in $n = 4$ dimensions refers to the supersymmetry group $P^{4|4}$, $N = 2$ supersymmetry to $P^{4|8}$, and $N = 4$ supersymmetry to $P^{4|16}$. This nomenclature is uneven in dimensions $n \equiv 2, 6 \pmod 8$ with chiral supersymmetry and is also not completely standard for $n = 1$. For example, $N = 1$ supersymmetry in $n = 2$ dimensions usually refers to $P^{2|(1,1)}$ whereas $N = 1$ supersymmetry in $n = 6$ dimensions usually refers to $P^{6|(8,0)}$.

There are inclusions of the supersymmetry groups obtained by setting some spatial translations to be trivial. Table 2 indicates sequences of algebras with a given number of supersymmetries (i.e., a given number of Q_a). These are the symmetry groups of supersymmetric field theories. The *classical* theories are often related by *dimensional reduction*, which is the process of reducing the dimension of a classical field theory by restricting to fields which are constant in a subspace of spatial directions. Notice that in dimensions $n \equiv 2, 6 \pmod 8$ we only obtain *nonchiral* theories by dimensional reduction, that is, theories with equal numbers of chiral supersymmetries. This is a well-known property of spin representations: No

# SUSYs	SUSY groups
1	$P^{1\ 1} \hookrightarrow P^{2\ (1,0)}$
2	$P^{1\|2} \hookrightarrow P^{2\|(1,1)} \hookrightarrow P^{3\|2}$
4	$P^{1\ 4} \hookrightarrow P^{2\ (2,2)} \hookrightarrow P^{3\|4} \hookrightarrow P^{4\|4}$
8	$P^{1\|8} \hookrightarrow P^{2\|(4,4)} \hookrightarrow P^{3\|8} \hookrightarrow P^{4\|8} \hookrightarrow P^{5\|8} \hookrightarrow P^{6\|(8,0)}$
16	$P^{1\ 16} \hookrightarrow P^{2\ (8,8)} \hookrightarrow \dots \hookrightarrow P^{6\|(8,8)} \hookrightarrow \dots \hookrightarrow P^{10\|(16,0)}$
32	$P^{1\|32} \hookrightarrow P^{2\|(16,16)} \hookrightarrow \dots \hookrightarrow P^{6\ (16,16)} \hookrightarrow \dots \hookrightarrow P^{10\|(16,16)} \hookrightarrow P^{11\ 32}$

Table 2: The supersymmetry groups

multiple of the half spin representation in dimension n extends to a representation of the spin group in higher dimensions.

We remark that in the context of dimensional reductions of classical field theories it is more proper to view Table 2 in terms of *subquotients* rather than inclusions. If we dimensionally reduce a theory on superspacetime M by considering fields constant along a spatial translation τ, then we may as well view fields as functions on $M/\mathbb{R}\tau$. If P is the super Poincaré group of M, the super Poincaré group of $M/\mathbb{R}\tau$ is the subquotient

$$(\text{normalizer of } \mathbb{R}\tau \text{ in } P) \ / \ \mathbb{R}\tau.$$

The form of the supersymmetry algebra is constrained by some physical properties of supersymmetric quantum field theories, as was deduced by Haag, Sohnius, and Lopuszański.

Central extensions and R-symmetry

There are two algebraic features of the supersymmetry groups which play an important role in some supersymmetric field theories: central extensions and outer automorphisms.

Recall that one important fact about the Lorentz signature is that in the symmetric square of a real spin representation we always find a copy of the vector representation. This is used to construct superspacetime (see (3.6)). For any real spin representation we decompose the symmetric square of the dual as

$$\operatorname{Sym}^2 S^* \cong V \ \oplus \ \mathbb{R}^c \ \oplus \bigoplus_i \Lambda^{p_i} V, \tag{3.12}$$

where $p_i \geq 1$. We can add the second term to the supersymmetry algebra (3.11):

$$\tilde{\mathfrak{p}} = \big(V \oplus \mathbb{R}^{c'} \oplus \mathfrak{so}(V)\big) \ \oplus \ S^*, \tag{3.13}$$

for some $c' \leq c$. We use the decomposition (3.12) to define the $\mathbb{R}^{c'}$ component in the bracket of odd elements and postulate that $\mathbb{R}^{c'}$ be central. It is easy to check that the Jacobi identity is satisfied. Thus $\tilde{\mathfrak{p}}$ is a central extension of $\mathfrak{p}$ and we have a corresponding central extension at the level of groups.

Elements of the central $\mathbb{R}^{c'}$ are called *central charges*. These central extensions arise in supersymmetric classical field theories. Recall that in any field theory (on Minkowski spacetime) the space $\mathcal{M}$ of classical solutions is a symplectic manifold. In a supersymmetric field theory there is an action of $\mathfrak{p}$ on $\mathcal{M}$ by infinitesimal symplectic transformations. When we lift these infinitesimal symmetries to functions, then we typically get a central extension by locally constant functions. These locally constant functions are the classical central charges. The Poisson brackets of the Noether charges and central charges obey an algebra opposite to that of (3.13).

We can also include any of the other terms in (3.12) in the even part of the superalgebra, but they are not central; they have nontrivial brackets with the Lorentz algebra $\mathfrak{so}(V)$. They play the role of central charges for *extended objects* in field theories, such as the "D-branes" which currently play a prominent role in string theory. (These extended supersymmetry algebras appeared earlier in the supergravity literature.) Although we will not meet them in these lectures, for reference we give the complete decomposition of $\mathrm{Sym}^2 S^*$ for the minimal S in Table 3. We also include two nonminimal examples in dimension $n = 10$; they correspond to two of the superstring (or supergravity) theories.

n	SUSY group based on S^*	$\mathrm{Sym}^2 S^*$
1	$P^{1\|1}$	V
2	$P^{2\|(1,0)}$	V^+
2	$P^{2\|(0,1)}$	V
3	$P^{3\ 2}$	V
4	$P^{4\ 4}$	$V \oplus \bigwedge^2 V$
5	$P^{5\|8}$	$V \oplus \mathbb{R} \oplus 3\bigwedge^2 V$
6	$P^{6\|(1,0)}$	$V \oplus 3\bigwedge^3_+ V$
6	$P^{6\|(0,1)}$	$V \oplus 3\bigwedge^3_- V$
7	$P^{7\|16}$	$V \oplus 3\mathbb{R} \oplus \bigwedge^2 V \oplus 3\bigwedge^3 V$
8	$P^{8\|16}$	$V \oplus 2\mathbb{R} \oplus \bigwedge^3 V \oplus \bigwedge^4 V$
9	$P^{9\|16}$	$V \oplus \mathbb{R} \oplus \bigwedge^4 V$
10	$P^{10\ (16,0)}$	$V \oplus \bigwedge^5_+ V$
10	$P^{10\|(0,16)}$	$V \oplus \bigwedge^5 V$
10	$P^{10\ (32,0)}$	$3V \oplus \bigwedge^3 V \oplus 3\bigwedge^5_+ V$
10	$P^{10\ (0,32)}$	$3V \oplus \bigwedge^3 V \oplus 3\bigwedge^5_- V$
10	$P^{10\|(16,16)}$	$2V \oplus \mathbb{R} \oplus \bigwedge^2 V \oplus 3\bigwedge^4 V \oplus \bigwedge^5 V$
11	$P^{11\|32}$	$V \oplus \bigwedge^2 V \oplus \bigwedge^5 V$

Table 3: Decomposition of $\mathrm{Sym}^2 S^*$

Example. We will encounter the classical version of this example in Lecture 4. Consider $P^{2|(1,1)}$, so the (self-dual) real spin representation $S^+ \oplus S^-$. Then

$$\mathrm{Sym}^2(S^+ \oplus S^-)^* \cong V^+ \oplus V^- \oplus \mathbb{R}.$$

(Recall that the vector representation $V \cong V^+ \oplus V^-$ is reducible in $n = 2$ dimensions.) Let $\partial_\pm$ be basis elements of $V^\pm$ and $Q_\pm$ basis elements of $(S^\pm)^*$. The nontrivial brackets in $\tilde{\mathfrak{p}}^{2|(1,1)}$ are

$$\begin{aligned} [Q_+, Q_+] &= -2\partial_+ \\ [Q_-, Q\ \ ] &= -2\partial_- \\ [Q_+, Q_-] &= 2Z, \end{aligned} \tag{3.14}$$

where Z is the central charge. Note that $\mathfrak{p}^{2|(1,1)}$ is obtained by setting $Z = 0$.

The group $P^{2|(1,1)}$ sits as a subgroup in $P^{3|2}$, as indicated in Table 2. In the larger group the bracket $[Q_+, Q_-]$ is -2 times infinitesimal translation in the spatial direction which has been set to zero in passing from $P^{3,2}$ to $P^{2|(1,1)}$. In the central extension $\tilde{P}^{2|(1,1)}$ this spatial translation has been set to minus the central element. More generally, for any $P^{n|s}$ which is obtained from a larger $P^{n'|s}$ by dimensional reduction, there is a central extension with $n' - n$ central charges. All of the central charges in Table 3 are accounted for by dimensional reduction, and it is possible that central charges always arise this way.

The connected group of outer automorphisms of the super Poincaré group which fix the Poincaré subgroup is called the R-symmetry group. In particular, R-symmetries only act nontrivially on the Q_a, not on the P_μ nor on the generators of Lorentz transformations. We will see an example when we study $P^{1|2}$ in Lecture 4 and $P^{4|4}$ in Lecture 5.

Representations of the super Poincaré group

Recall from Lecture 1 that the Hilbert space of a quantum field theory in n dimensions is a nonnegative energy unitary representation of the Poincaré group $\check{P}^n$. Quantum particles are irreducible representations. In a supersymmetric theory this representation extends to a unitary representation of $P^{n,s}$ for some s depending on the amount of supersymmetry. The irreducible representations are not irreducible when restricted to the subgroup $\check{P}^n$, but rather break up as a finite sum. Such collections of particles are called supersymmetric *multiplets*. The particle content of a supersymmetric theory is organized into such multiplets. There is a general story, but to understand supersymmetric quantum field theories one needs to learn the taxonomy of multiplets in various dimensions with various amounts of supersymmetry. We begin with the general theory and work through the examples relevant to Lecture 4. In Lecture 5 we will return to compute more examples which are relevant there.

Recall that in interacting theories the spins of massless particles are constrained to be at most 2, and if there is no gravity then at most 1. Werner Nahm observed that this severely constrains the possible supersymmetric theories, and this condition is the reason why Table 1 stops at $n = 11$. We will see later in this lecture how the quantization of free classical fields leads to irreducible representations of $\check{P}^n$.

Thus it is natural to use the analysis of supersymmetry particle multiplets to constrain and predict (nonfree) supersymmetric classical field theories. For example, in this way Nahm predicted some supergravity theories which were constructed later.

Consider the supersymmetry group $P^{n|s}$. (We understand $s = (s^+, s^-)$ if $n \equiv 2, 6 \pmod 8$.) As in (1.2) we consider the subgroup V of translations:

$$1 \to V \to P^{n|s} \to \mathrm{Spin}(1, n-1) \ltimes \Pi S^* \to 1$$

Suppose we are given an irreducible positive energy representation of $P^{n|s}$. Upon restriction to V we obtain an irreducible orbit of infinitesimal characters of some mass $m \geq 0$ in V^*, as in Figure 2. The little groups

$$\begin{aligned} m > 0: &\qquad \mathrm{Spin}(n-1) \ltimes \Pi S^* \\ m = 0: &\qquad \mathrm{Spin}(n-2) \ltimes \Pi S^* \end{aligned} \tag{3.15}$$

now include odd parts as well. (Recall that the little group is the reductive part of the stabilizer.) The representation of $P^{n|s}$ is induced from a projective graded finite dimensional unitary representation of the little group—as before, we form a homogeneous graded hermitian vector bundle over the orbit and take the Hilbert space of sections. The fact that the representation is projective is due to the nontrivial bracket on S^*. More precisely, at $\lambda \in V^*$ we seek a representation of the central extension constructed using the quadratic form q_λ on S^* given by

$$q_\lambda(Q_1, Q_2) = \lambda([Q_1, Q_2]).$$

Recall the positivity condition (3.7). Since λ is in the closure of the positive dual lightcone, then in view of (3.9) we conclude that q_λ is negative semidefinite. In the massive case the form is negative definite, whereas in the massless case it has a kernel. Except for the exceptional case $s = 1$ the kernel has dimension $s/2$. (This follows by splitting V into the sum of a two-dimensional Lorentz space and a negative definite complement, thus reducing to the $n = 2$-dimensional case and an explicit computation.) In the massless case we work with the quotient of S^* by the kernel, and then in both cases we obtain a negative definite space. The projective representations of (3.15) we want are then graded Clifford modules

$$W = W^0 \oplus W^1$$

for S^* (modulo the kernel in the massless case) together with an intertwining action of the appropriate Spin group. Our sign conventions for graded Hilbert spaces and self-adjoint operators lead to a slightly different construction than usual.

A few remarks before the examples:

- In this construction we end up considering spinors of spinors—a Clifford module for the vector space of spinors. Also, keep in mind that S is a *real* space, whereas the representation of the little group we construct -the spinors of spinors—is *complex*. The space S is a representation of $\mathrm{Spin}(n-1)$ or $\mathrm{Spin}(n-2)$ by restriction of the $\mathrm{Spin}(1, n-1)$ action.
- Usually $\dim S$ is even and we construct the unique irreducible graded Clifford module by complexifying S and choosing a lagrangian splitting, as described in (1.20).

- The irreducible massless representations tend to be smaller than the irreducible massive representations because of the drop in dimension. We will not see that in our example below since s is very small.
- The representation theory of the supersymmetry groups with central charges is similar. We present it through a simple example below. The important concept of a *BPS particle* is introduced there.
- Allowable representations in a local quantum field theory are constrained to be CPT invariant, as described in Lecture 1.
- We choose the grading of W—which piece is even and which piece odd—according to the spin-statistics principle.
- The even and odd pieces of a Clifford module have equal dimensions. This means that *a supersymmetry multiplet has an equal number of bosonic degrees of freedom and fermionic degrees of freedom.* (The number of degrees of freedom is the dimension of the representation of the little group.) There is an exception for chiral theories in $n = 2$, however. For example, there exists a theory with symmetry group $P^{2|(1,0)}$ with one (chiral) fermion and no bosonic degrees of freedom.

Example 1 ($P^{3|2}$). We first review the representations of the group $\check{P}^3$. For massive representations the little group is Spin(2), and its irreducible complex representations are one-dimensional and labeled by the spin j. All of the representations are CPT invariant. The spin $j = 0$ representation is called a *massive scalar particle*, the spin $j = 1/2$, $j = -1/2$ representations *massive spinor particles*, and the spin $j = 1$, $j = -1$ representations *massive half-vector particles*. (The *massive vector particle* is the sum of the $j = 1$ and $j = -1$ representations.) In the massless case the little group is $\mathrm{Spin}(1) \cong \mathbb{Z}/2$. There are two irreducible complex representations, each of dimension one. The trivial one is the *massless scalar particle* and the nontrivial one is the *massless spinor particle*.

Why didn't we mention the massless vector particle? We'll answer that question at the end of the lecture.

Now we turn to the supersymmetry group $P^{3|2}$. In the massive case we have in addition to the little group Spin(2) a negative definite Clifford algebra generated by S^*, which here has dimension 2. After complexification we choose a basis $\{Q_+, Q_-\}$ where Spin(2) acts on $Q_\pm$ by $j = \pm 1/2$. A physicist describes the Clifford module as follows. Fix a *vacuum* vector $|0\rangle$ and postulate

$$Q_-|0\rangle = 0.$$

Then the Clifford module is

$$W = \mathbb{C} \cdot |0\rangle \oplus \mathbb{C} \cdot Q_+|0\rangle.$$

To define the action of Spin(2) we need to specify a spin j for $|0\rangle$; then the spin of $Q_+|0\rangle$ is $j + 1/2$.

To summarize: A massive multiplet for $P^{3|2}$ is a pair of particles of adjacent spins j and $j + 1/2$.

We show some possibilities in Table 4, which indicates the number of representations with a given spin j which occur in each multiplet. We only list multiplets with spin at most 1; they are the relevant multiplets for supersymmetric theories

	-1	$-1/2$	0	$1/2$	1
massive scalar multiplet		1	1		
massive scalar multiplet			1	1	
massive vector multiplet	1	1		1	1

Table 4: Massive multiplets for $P^{3\,2}$

without gravity. The multiplets are usually named after the highest spin boson. There are two *massive scalar multiplets* because of the two different possibilities for a massive spinor. In addition to the *massive vector multiplet* shown, there are also *massive half-vector multiplets.*

At the end of this lecture we will learn how to construct representations from fields. It is instructive to see how the two different spinor particles arise—the difference comes from the sign in the mass term in the lagrangian. We will not consider massive vectors in these lectures, but it is amusing to see how the massive half-vectors can be written in terms of fields—the 3-dimensional Chern-Simons term enters.

Now we consider the massless multiplets. In this case the quadratic form on S^* is degenerate, and the quotient by the kernel is one-dimensional. The irreducible complex Clifford module $W = W^0 \oplus W^1$ is still two-dimensional. The little group $\mathrm{Spin}(1) \cong \mathbb{Z}/2$ acts trivially on W^0 and nontrivially on W^1. So there is a unique massless multiplet, the *massless scalar multiplet,* which contains one scalar particle and one spinor particle.

Example 2 ($\tilde{P}^{2|(1,1)}$). This example illustrates the representation theory for a centrally extended supersymmetry group. We refer to (3.14) for the bracketing relations in the central extension $\tilde{\mathfrak{p}}^{2|(1,1)}$ of the symmetry algebra $\mathfrak{p}^{2|(1,1)}$. From these brackets we easily compute

$$\frac{1}{4}[Q_+ \pm Q_- \,,\, Q_+ \pm Q_-] = -\partial_t \pm Z, \tag{3.16}$$

where

$$2\partial_t = \partial_+ + \partial_-$$

is infinitesimal time translation. Note that the left hand side of (3.16) is the square of an odd element. Thus the quantum operators $\hat{H} \pm \hat{Z}$ corresponding to the right hand side of (3.16) are nonnegative; in other words,

$$\hat{H} \geq |\hat{Z}|. \tag{3.17}$$

Here $\hat{H}$ is the quantum hamiltonian. In an irreducible supersymmetry representation the operator $\hat{Z}$ is a constant since Z is central. For $Z = 0$ the inequality (3.17) is a special case of the general argument that the quantum hamiltonian in a supersymmetric theory is nonnegative. But with the central charge we have a stronger bound. The Poincaré invariant statement is the *BPS bound*: the *mass* is bounded below by one-half the absolute value of the central charge. Furthermore, if we have

equality—the mass equal to one-half the absolute value of the central charge—then the quadratic form on the odd part of the little group has a half-dimensional kernel, just as for massless representations. (In this case that is evident in (3.16).) So we obtain special irreducible massive multiplets—*BPS multiplets*—which typically have fewer degrees of freedom than the usual massive multiplets. If this is the case, then they are stable under perturbations. Hence these BPS representations are an important source of stable particles in supersymmetric theories.

The states in a BPS representation are annihilated by 1/2 of the supersymmetry, in this example by $Q_+ + Q_-$ or $Q_+ - Q_-$. There are more complicated situations where a different fraction of the supersymmetry annihilates a representation, which is also termed BPS.

In Lecture 4 we will see a classical field configuration which satisfies a classical version of the BPS condition for the supersymmetry group $\tilde{P}^{2|(1,1)}$: it solves the equations of motion and is annihilated by half of the classical supersymmetry.

A quick tour of fields and classical field theories

We restrict the discussion to field theories formulated on Minkowski spacetime $\check{M} = \check{M}^n$. Then a field is a function (or tensor field or spinor field—in general, a section of a fiber bundle) on $\check{M}$. In our discussion and in all examples of field theories we consider we assume the spin-statistics connection. For a classical field the *spin* is determined by the representation of the Lorentz group under which it transforms. Thus differential forms have integer spin whereas spinor fields have half-integer spin. Spin-statistics asserts that fields of integer spin are classical bosons, whereas fields of half-integer spin are classical fermions. The space of fields is a supermanifold whose underlying manifold of real points consists of classical bosons.

In Minkowski spacetime the space of fields is *real* and the lagrangian of a unitary field theory is also *real*. The analytic continuation, or Wick rotation, to Euclidean space—which we do not consider in these lectures—may not be real.

The catalog of fields which arise is brief, and it is helpful to survey it briefly if for no other reason than to establish vocabulary. We proceed according to spin, beginning with fields of integer spin—bosons. Here we discuss both linear and nonlinear fields. For linear fields we have the possibility of writing free (noninteracting) lagrangians, and we will do that in the next section. Nonlinear fields are necessarily interacting.

The simplest *spin 0* field is a *real scalar field* $\phi\colon \check{M} \to \mathbb{R}$. The space of all real scalar fields is a real vector space. Quite generally, a *scalar field* is a map $\phi\colon M \to X$ for some manifold X. To write down a kinetic term we need a Riemannian metric on X.

One example of a linear *spin 1* field is a 1-form $\alpha \in \Omega^1(\check{M})$. The space of these fields is again a real vector space. We can take instead the space of connections with abelian structure group $\mathbb{R}^{>0}$ or $\mathbb{T}$, the circle group. If we fix a principal bundle P with one of these structure groups, then the space of connections on P is a real affine space whose associated vector space of translations is isomorphic to the space of 1-forms. A connection is also called a *gauge field*. These theories have *gauge equivalence* or *local symmetry*: we work on the quotient of the space of fields by gauge transformations. For 1-forms this means that the space of fields is the real vector space $\Omega^1(\check{M})/d\Omega^0(\check{M})$. The "space" of abelian connections is really a category—already the bundles on which the connections live form a category—and

the space of connections modulo gauge equivalence is the affine space of equivalence classes in the category. There is a nonlinear version of these spin 1 fields: the space of gauge equivalence classes of connections with some nonabelian structure group G. There are other *linear* spin 1 fields as well, namely p-form fields with $1 < p < n$. Again there is a gauge equivalence and the true space of fields is the quotient $\Omega^p(\check{M})/d\Omega^{p-1}(\check{M})$. There is also an affine version analogous to abelian connections, but it is more difficult to describe. However, there is not a known nonlinear or nonabelian analog of p-form fields. The p-form fields with $p \geq 2$ often arise in supergravity theories in dimensions $n \geq 6$. We will not encounter them in these lectures.

Finally, a *spin 2* field is a traceless symmetric tensor field or a metric. It is the basic field in theories of gravity. Here too there is local symmetry: the group of local symmetries is the diffeomorphism group.

Turning now to fermions, or fields of half-integral spin, a *spin 1/2* field is a *spinor field* $\psi\colon \check{M} \to S$ for some real spin representation S of $\mathrm{Spin}(1, n-1)$. This is the second use for the information in Table 1; the first was in the construction of superspacetime. Here too it is important that we use *real* spin representations, since fields in Minkowski spacetime are real. The representation S need not be minimal. There is no nonlinear version of fermion fields.

A *spin 3/2* field is called a *Rarita-Schwinger field*. It is a function $\chi\colon \check{M} \to (S \otimes V^*)_0$, where S is a minimal real spin representation and the subscript means that we take the highest weight irreducible piece of the tensor product. These fields only arise in supergravity theories. There is a local symmetry as well: we work on the quotient by the image of the covariant derivative of spinor fields $\check{M} \to S$.

To describe a field theory in Minkowski spacetime we might first list the field content: scalar fields with values in such-and-such a manifold, a gauge field with gauge group so-and-so, spinor fields of this-or-that particular type, and so on. A *bosonic theory* is one whose fields are all bosonic. There is a standard classification of theories according to the field content. A bosonic theory with only scalar fields is called a *σ-model*. If the scalar fields take values in a linear space it is called a *linear σ-model*, otherwise a *nonlinear σ-model*. A *bosonic gauge theory* includes a gauge field (connection) and possibly also scalar fields. A *pure* gauge theory is a theory with only a gauge field; theories which also include scalar fields can be described as the *coupling* of a gauge theory with a σ-model. One form of coupling is that the scalar field might be a section of a fiber bundle associated to the connection field. For example, we might have a theory with an abelian connection, say with structure group $\mathbb{T}$, and a complex scalar field which is a section of an associated complex line bundle. A bosonic theory of *gravity* has the spin 2 metric and then possibly gauge fields and scalar fields as well.

We can also construct theories with fermions only or with both bosons and fermions. The theories we consider in these lectures are *supersymmetric*; a supersymmetry group acts as a group of symmetries of the classical theory. Since supersymmetries exchange even and odd fields, such theories necessarily have both bosons and fermions (with some exotic exceptions in dimension 2). The theories are named after the highest spin boson field. For example, a *supersymmetric σ-model* contains only scalar fields and spinor fields. A *supersymmetric gauge theory* contains gauge fields and spinor fields; it may contain scalar fields as well. Finally, a *supergravity* theory contains a metric and Rarita-Schwinger field, and it may contain fields of lower spin as well. For example, the two supergravity theories in

$n = 10$ dimensions with $s = 32$ supersymmetries contain fields of all the different possible spins.

A supersymmetric theory with more than the minimal number of supersymmetries is often said to have *extended supersymmetry*, and the corresponding supersymmetry groups are referred to as *extended supersymmetry groups*. These theories often, but not always, come by *dimensional reduction* from a theory in a higher spacetime dimension with the same number of supersymmetries. (We dimensionally reduce a classical theory by restricting to fields which are constant along some spatial directions. We give some examples in Lecture 4.) Theories with minimal supersymmetry are simpler, and if a theory with extended supersymmetry comes by dimensional reduction from one with minimal supersymmetry, then it is easier to understand that way. There are theories with extended supersymmetry which are *not* dimensional reductions, most notably theories with extended *chiral* supersymmetry in dimensions $n \equiv 2, 6, 10 \pmod 8$. (These theories have supersymmetry group $P^{n|(s^+,s^-)}$ and more than the minimal amount of supersymmetry.)

# SUSY	maximal dimension	σ-model	gauge theory	supergravity
1	2	S	S	S
2	3	S	S	S
4	4	S (Kähler)	S	S
8	6	✓ (hyperkähler)	S	✓
16	10		✓	✓
32	11			✓

Table 5: Supersymmetric theories

In Table 5 we catalog supersymmetric theories according to the number of supersymmetries. In the second column we indicate the maximal spacetime dimension n in which the theory may be formulated; theories in lower dimension are obtained by dimensional reduction. Keep in mind that there are exceptional theories with extended supersymmetry—like the chiral theories just mentioned—which do not come by dimensional reduction. The chiral theories in dimensions 2,6,10 listed in Table 5 have the minimal amount of supersymmetry. In the maximal dimension each theory listed in the table has minimal supersymmetry. An 'S' indicates that there is a superspacetime formulation; a '✓' indicates that the theory exists, but there is no adequate superspacetime formulation (off-shell); and a blank spot in the table means that there is no theory. The blank spots can be predicted from the representation theory of the supersymmetry group and the constraint on the spin of massless interacting quantum particles, as we indicated in the last section. A few remarks about the table:

- Not only are there special theories not obtained by dimensional reduction, but even in theories which are dimensional reductions there are sometimes terms one can add to the lagrangian which do not come from the higher dimensional theory.
- In Lecture 2 we discussed two formulations of the superparticle. First, we described the theory in *components*, that is, in terms of ordinary fields.

Then we gave a super(space)time formulation in terms of *superfields*. All theories with a superspacetime formulation (marked 'S' in Table 5) may be described in terms of either superfields or in terms of component fields. Theories with no superspacetime formulation only have a description in terms of "component fields", though these fields are not the components of anything—they are just fields on ordinary Minkowski spacetime. We will see that the component formulation of theories which have a superspacetime formulation often includes *auxiliary fields* which enter the lagrangian algebraically.

- Theories with a superspacetime formulation have manifest supersymmetry in the superspacetime formulation. The supersymmetry algebra closes off-shell. In the component formulation the supersymmetry is not manifest, but the algebra still closes off-shell. (For that we need to include the auxiliary fields if there are any.) If there is no superspacetime formulation, then the supersymmetry algebra does not close off-shell. (Recall the example of the coupled harmonic oscillators in Lecture 2 for an analog of these phenomena in ordinary classical mechanics.)
- The superspacetime formulations are most useful for understanding the supersymmetry, since in this way it is manifest. However, to see the physics of the theory it is often best to work in components. Some computations are easier in superspacetime, some easier in components. If there is a superspacetime formulation, then it is useful in the quantum theory to give *a priori* constraints on the possible quantum corrections. Such corrections must respect the supersymmetry, and the constraints imposed are more easily seen in superspacetime.
- The most commonly used superspacetimes are those in dimensions $n \leq 4$ with $s \leq 4$ supersymmetries. For theories with more supersymmetry in dimensions $n \leq 4$, we can still use the $s = 4$ superspacetime to keep part of the supersymmetry manifest. Nevertheless, superspacetimes with $s > 4$ have striking applications.
- We indicated that there is a superspacetime formulation of supersymmetric gauge theories with 8 supersymmetries, but this only applies to *pure* gauge theories. The superspacetime formulation in 6 dimensions is a bit deficient in some ways; the reduction to 4 dimensions is better.
- There are superspacetime formulations of σ-models with 8 supersymmetries for special kinds of hyperkähler manifolds, but none which works in general.

From classical fields to quantum particles

We indicate how to pass from classical free fields on Minkowski spacetime $\check{M}^n$ to positive energy unitary representations of the symmetry group $\check{P}^n$. A lagrangian is *free* if it leads to a linear or affine symplectic space of classical solutions with a constant symplectic form. For a free supersymmetric theory the construction we describe gives a representation of the appropriate supersymmetry group. We restrict our discussion to real scalar fields, spinor fields, and abelian connections. The story for other free fields (p-form fields, Rarita-Schwinger fields, and symmetric tensor fields) is similar. We leave the easy computations to the reader as exercises in the classical field theory yoga described in the previous lecture.

Let $\check{M} = \check{M}^n$ denote n-dimensional Minkowski spacetime. We take standard coordinates $x^0 = ct, x^1, \dots, x^{n-1}$, where c is the speed of light and t represents time. The metric is

$$(dx^0)^2 - (dx^1)^2 - \cdots - (dx^{n-1})^2$$

and the standard density is

$$|d^n x| = |dx^0\, dx^1 \cdots dx^{n-1}|.$$

The wave operator on functions is

$$\Box = (-1)^{n-1} * d * d = -d^* d = \frac{\partial^2}{(\partial x^0)^2} - \frac{\partial^2}{(\partial x^1)^2} - \cdots - \frac{\partial^2}{(\partial x^{n-1})^2}.$$

Example 1 (free real scalar field). Let $\phi\colon \check{M} \to \mathbb{R}$ be a real scalar field. We take as lagrangian density

$$L = \Big(\frac{1}{2}|d\phi|^2 - \frac{m^2}{2}\phi^2\Big)\,|d^n x|. \tag{3.18}$$

Here $m \geq 0$ is the mass of the field. (There is a factor of $c/\hbar$ which is necessary to make the units come out correctly; it is set to one in *natural units* $\hbar = c = 1$.) Then a short computation shows that the variational 1-form is

$$\gamma = \delta\phi \wedge *d\phi$$

and the local symplectic form is

$$\omega = *d\delta\phi \wedge \delta\phi. \tag{3.19}$$

The equation of motion which defines the space of classical solutions $\mathcal{M}$ is the linear wave equation

$$(\Box + m^2)\phi = 0. \tag{3.20}$$

The space of fields $\mathcal{F}$ is a real vector space, and the space of classical solutions $\mathcal{M}$ to (3.20) is a linear subspace. It carries a symplectic form which we obtain by integrating (3.19) over a codimension one spacelike subspace of Minkowski spacetime.

We remark that for this integral to converge we need to impose some L^2 conditions on the spatial integrals of fields and their variations.

We analyze equation (3.20) by fixing an origin in $\check{M}$ and passing to the Fourier transform $\hat{\phi}\colon V^* \to \mathbb{C}$. It is a complex function which satisfies the reality condition

$$\hat{\phi}(-k) = \overline{\hat{\phi}(k)}, \qquad k \in V^*. \tag{3.21}$$

The classical equation of motion (3.20) is

$$\big(-|k|^2 + m^2\big)\hat{\phi}(k) = 0,$$

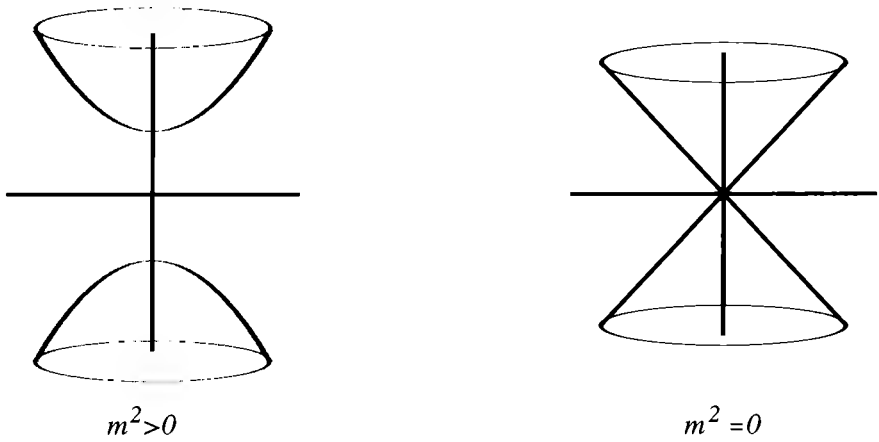

Figure 7: The mass shell for $m^2 > 0$ and $m^2 = 0$

and so solutions are supported on the mass shell $\{k \in V^* : |k|^2 = m^2\}$ depicted in Figure 7. Note that these solutions form a real representation of the symmetry group $\check{P}^n$.

We quantize $\mathcal{M}$ just as we did the free particle in Lecture 1. That is, we choose a polarization and take the quantum Hilbert space to be the space of functions on one of the lagrangian subspaces in the polarization. In this example we choose a *complex* polarization

$$\mathcal{M} \otimes \mathbb{C} \cong L \oplus \overline{L}.$$

The complexification of $\mathcal{M}$ consists of complex functions on the mass shell with no reality condition, and we take L to be the subspace of functions supported on the set of points with positive time. Then the quantum Hilbert space is the Hilbert space completion of the Fock space:

$$\mathcal{H} = L^2(L) \supset \mathrm{Sym}^\bullet(L^*).$$

The *one-particle Hilbert space* is

$$\mathcal{H}_1 = L^*.$$

It is the irreducible representation of $\check{P}^n$ with the trivial spin 0 representation of the little group. This is the quantum particle which corresponds to the free field ϕ. States in the nth symmetric power of L^* contain n indistinguishable particles.

Observe that in the massless case there is a global symmetry group of translations in the codomain $\mathbb{R}$ of the scalar field.

Example 2 (free spinor field). Let S be a real spin representation of the Lorentz group $\mathrm{Spin}(1, n-1)$. A spinor field is a map $\psi\colon \check{M} \to \Pi S$. In this case the space of fields $\mathcal{F}$ is an odd real vector space. We use the symmetric pairing

$$\tilde{\Gamma}\colon S \otimes S \longrightarrow V$$

(see (3.2)) to construct the kinetic term. A mass term exists only if we have a nonzero *skew-symmetric* pairing

$$M\colon S \otimes S \longrightarrow \mathbb{R}.$$

If there is a nonzero mass pairing M, then there exists a *normalized* skew-symmetric pairing

$$\epsilon\colon S \otimes S \longrightarrow \mathbb{R};$$

the normalization condition is (3.3) given (3.5). For simplicity assume that ϵ is irreducible. Then we can write

$$M = m\epsilon, \qquad m \in \mathbb{R}. \tag{3.22}$$

The free lagrangian density for the spinor field is

$$L = \left(\frac{1}{2}\psi \not{D} \psi - \frac{1}{2}\psi M \psi\right) |d^n x|. \tag{3.23}$$

The kinetic term is the *Dirac* form

$$\psi \not{D} \psi = \tilde{\Gamma}^\mu(\psi, \partial_\mu \psi) = \tilde{\Gamma}^{\mu\alpha\beta} \psi_\alpha \, \partial_\mu \psi_\beta.$$

It is a symmetric (in the graded sense) bilinear form up to an exact term, so it is exactly symmetric after integrating over M, assuming no contribution at infinity. We use the convention that the name of a bilinear form is written between the arguments. The mass pairing is

$$\psi M \psi = M^{\alpha\beta} \psi_\alpha \psi_\beta.$$

There is a similar lagrangian for *dual spinor fields* $\lambda\colon \check{M} \to \Pi S^*$.

The equation of motion asserts that ψ is in the kernel of $\not{D} - M$. As for the scalar field, we analyze this condition using the Fourier transform. Then ψ is a classical solution if and only if

$$\sqrt{-1}\, \tilde{\Gamma}^{\mu\alpha\beta} k_\mu \hat{\psi}_\beta(k) = M^{\alpha\beta} \hat{\psi}_\beta(k), \qquad \text{for all } \alpha. \tag{3.24}$$

Multiplying both sides by $\sqrt{-1}\, \Gamma^\nu_{\gamma\alpha} k_\nu$, we deduce from the Clifford relation (3.3) and also (3.22), (3.5) that $\hat{\psi}$ is supported on the mass shell $|k|^2 = m^2$. Furthermore, (3.24) imposes a constraint on $\hat{\psi}$ restricted to the mass shell; it cuts down by a factor of 2 the number of degrees of freedom. Therefore, the space of classical solutions $\mathcal{M}$ to the Dirac equation is the space of "real" sections of an odd complex vector bundle E of rank $\dim S/2$ over the mass shell; the reality condition (3.21) is that coming from the Fourier transform. As before we quantize using a complex polarization. The one-particle (odd) Hilbert space is the space of sections of E^* on the "forward half" of the mass shell. It is a spin 1/2 representation of $\check{P}^n$ with $\dim S/2$ degrees of freedom. The (graded) Hilbert space of the free quantum theory is the completed symmetric algebra on the one-particle Hilbert space; it contains both fermions and bosons.

Example 3 (massless abelian gauge field). For definiteness take the gauge group to be the circle group $\mathbb{T}$ of unit norm complex numbers. On the Lie algebra $\mathrm{Lie}(\mathbb{T}) \cong \sqrt{-1}\,\mathbb{R}$ we have the positive definite inner product $\langle a, b\rangle = -ab$. The field in a gauge theory is a connection A on $\check{M}$ with structure group $\mathbb{T}$, and we work with fields up to gauge equivalence. The space of equivalence classes is a real affine space. The lagrangian is

$$L = -\frac{1}{2}\langle F_A \wedge *F_A\rangle = -\frac{1}{2}|F_A|^2\,|d^n x|. \tag{3.25}$$

The factor $-\frac{1}{2}$ is chosen so that the kinetic term is properly normalized—write (3.25) out in coordinates to check! The classical equation of motion is the linear equation

$$d * F_A = 0 \tag{3.26}$$

on the curvature, and the classical moduli space of connections modulo equivalence which satisfy (3.26) is an affine subspace of the space of fields mod equivalence. We can quantize this symplectic affine space as usual.

Of course, the detailed analysis involves choosing an origin, so we may as well from the beginning work with the vector space of translations associated to connections. Hence we take the field to be a *real* 1-form α up to addition of an exact real 1-form. The lagrangian is

$$L = -\frac{1}{2}d\alpha \wedge *d\alpha = -\frac{1}{2}|d\alpha|^2\,|d^n x|,$$

and the equation of motion is

$$d * d\alpha = 0.$$

As usual, we consider the Fourier transform $\hat{\alpha} = \hat{\alpha}_\mu dx^\mu$ on V^*. The vector space of classical solutions is the first cohomology space of the complex

$$\Omega^0(\check{M}) \xrightarrow{d} \Omega^1(\check{M}) \xrightarrow{*d*d} \Omega^1(\check{M}).$$

On the Fourier transforms we have

$$\begin{aligned} d\hat{f}(k) &= \sqrt{-1}\,\hat{f}(k)\,k \\ *\,d\,*\,d\hat{\alpha}(k) &= |k|^2\hat{\alpha}(k) - \langle\hat{\alpha}(k), k\rangle k. \end{aligned}$$

An easy argument shows that the first cohomology is isomorphic to the space of "real" functions $\{\hat{\alpha}\colon \text{lightcone} \to V^*\otimes\mathbb{C}\}$ on the lightcone which satisfy $\langle\hat{\alpha}(k), k\rangle = 0$ modulo the space of functions $\{\hat{\alpha}(k) = \hat{f}(k)k\}$, where $\hat{f}$ ranges over the complex functions on the lightcone. (Again the reality condition is that of the Fourier transform.) Again we polarize by choosing the lagrangian subspace of complex-valued functions supported on the forward lightcone. Its dual, the one-particle Hilbert space, is the irreducible representation of $\check{P}^n$ induced from the $(n-2)$-dimensional representation of the massless little group $\mathrm{Spin}(n-2)$, the so-called "vector representation". It has $n-2$ degrees of freedom. The Hilbert space of the free theory is the completed symmetric algebra on this representation.

Finally, we answer the question raised earlier: What is special about a massless vector particle in $n = 3$ dimensions? Recall that the little group is $\mathrm{Spin}(1) \cong \mathbb{Z}/2$

and there is only one bosonic representation—the trivial representation. It is the quantum particle which corresponds to the free massless scalar field on $\check{M}^3$. But clearly we can also consider a free massless abelian gauge field on $\check{M}^3$. From Example 3 we see that the corresponding quantum particle is the same as for the massless scalar field. This is the first example of a general and important phenomenon called *duality*. A single quantum theory—here the free field theory of a massless boson in $n = 3$ dimensions—has distinct realizations in terms of fields.

LECTURE 4
Theories with Two Supersymmetries

To date we have discussed many generalities about field theories and supersymmetry, but we have really only seen one supersymmetric theory—the superparticle. In this lecture we study in detail the classical supersymmetric nonlinear σ-model with two supersymmetries, and at the end of the lecture we briefly discuss supersymmetric gauge theory with two supersymmetries. These models have minimal supersymmetry in $n = 3$ dimensions (see Table 5) and can be dimensionally reduced to $n = 2$ and $n = 1$ dimensions.

At the end of Lecture 3 we introduced the free field theories—both classical and quantum for scalar fields, spinor fields, and connections. Here we begin with a general discussion of nonfree bosonic classical theories: nonlinear σ-models and nonabelian gauge theories. We introduce some terminology and important basic concepts from physics. More general lagrangians are discussed at the beginning of Lecture 5. As another preliminary we introduce coordinates for the superspacetime $M^{3|2}$ on which these models are defined.

We define the supersymmetric σ-model with target a general Riemannian manifold X directly in superspacetime. Physicists call this the *superfield formulation* or *superspace formulation*, since the fields in this approach—called *superfields*—are functions on superspacetime. The superfield formulation has the advantage that the supersymmetry is manifest and closes off-shell. However, it is not as immediate to read off the basic physics in these terms, and so we derive the *component field formulation* of the theory. Here the fields are defined on ordinary Minkowski spacetime and the lagrangian can be analyzed according to our general discussion. We also introduce a potential energy term in terms of a function $h\colon X \to \mathbb{R}$ and analyze that theory as well. Upon reduction to $n = 2$ dimensions we meet a new phenomenon—solitons—here derived using the BPS condition. Finally, we briefly discuss the reduction to mechanics, which is a superparticle with two supersymmetries.

In the superfield formulation of pure supersymmetric gauge theory in $M^{3|2}$ we meet *constrained* superfields for the first time. Constraints play a crucial role in theories with more supersymmetry and in supergravity.

σ-models and gauge theories

Throughout, we let $\check{M}^n$ denote n-dimensional Minkowski spacetime. We fix standard coordinates $x^0 = ct,\ x^1,\ \dots,\ x^{n-1}$ and the metric

$$(dx^0)^2 - (dx^1)^2 - \cdots - (dx^{n-1})^2. \tag{4.1}$$

Here c is the speed of light and t is time. In the sequel we work in natural units where $c = 1$, and so then $x^0 = t$. Let

$$|d^n x| = |dx^0 dx^1 \dots dx^{n-1}|$$

be the standard density on $\check{M}^n$. Using the splitting into time and space we obtain a density

$$|d^{n-1}x| = |dx^1 \dots dx^{n-1}|$$

on spacelike slices $x^0 = \text{const}$.

In this simplified exposition we only discuss *fundamental* lagrangians. The distinction between these and *effective* lagrangians is discussed at the beginning of Lecture 5. Also, we omit special features in low dimensions (such as dualities, special terms like Chern-Simons, ...).

Fix a Riemannian manifold X. Then the field in a σ-model is a map

$$\phi\colon \check{M}^n \longrightarrow X.$$

The other piece of data is a *potential energy* function

$$V\colon X \longrightarrow \mathbb{R}.$$

The fundamental lagrangian for the model is

$$L = \left\{ \frac{1}{2}|d\phi|^2 - \phi^* V \right\} |d^n x|. \tag{4.2}$$

The classical solutions obey the higher dimensional version of Newton's law:

$$\Box\phi = -\phi^* \operatorname{grad} V. \tag{4.3}$$

In the free case, when X is an affine space and V a quadratic function, the space of solutions is a an infinite dimensional symplectic affine space. In general, the space of solutions is nonlinear.

Of special interest are solutions of minimal *energy*, which comprise the *moduli space of vacua* $\mathcal{M}_{\text{vac}}$. This is sometimes termed the *classical moduli space*, though one has to be careful to distinguish from the moduli space of *all* solutions to (4.3). The energy is computed by integrating minus the Noether current for infinitesimal time translation over a spacelike codimension one submanifold of $\check{M}^n$. For simplicity we take that manifold to be $\{x^0 = \text{const}\}$; then the energy density is

$$\mathcal{E} = \left\{ \frac{1}{2}\left(\frac{\partial\phi}{\partial x^0}\right)^2 + \sum_{i=1}^{n-1} \frac{1}{2}\left(\frac{\partial\phi}{\partial x^i}\right)^2 + V(\phi) \right\} |d^{n-1}x|. \tag{4.4}$$

On the space of classical solutions the energy E, which is the integral of (4.4) over the manifold $x^0 = \text{const}$, is independent of the constant time, since energy is a conserved Noether charge (for minus infinitesimal time translation). The density in (4.4) is *not* Poincaré-invariant; it is just one piece of the Poincaré-invariant energy-momentum tensor. From (4.4) we see that a scalar field has minimal energy if it is static and constant in space, with constant value at a minimum of V. Assume that V is bounded below and shift the minimum value of V to be 0. (Recall that in supersymmetric theories we necessarily have $V \geq 0$.) Therefore,

$$\mathcal{M}_{\text{vac}} = V^{-1}(0).$$

Notice that any vacuum solution is obviously Poincaré-invariant.

We can consider local minima of V as vacua also, but in that case we need to adjust V by a constant to make V vanish at the vacuum we are considering. Note that we are free to redefine the energy by a constant, but we must do so in a Poincaré-invariant fashion by modifying the whole energy-momentum tensor.

The group H of *global* symmetries of the σ-model is the group of isometries of X which preserve V. They act as manifest symmetries of the theory. In particular, they preserve the moduli space of vacua $\mathcal{M}_{\text{vac}}$. Now a given vacuum ϕ_0 may or may not be fixed by H. Physicists say that the group H is *spontaneously broken* to the subgroup $\text{Stab}_H\, \phi_0$. For example, in any vacuum the Poincaré symmetry is *unbroken*.

We set up perturbation theory by expanding around a given vacuum ϕ_0. We simply write ϕ as a Taylor series around ϕ_0. Then the potential V has no linear term, since we are at a critical point. The quadratic term, which is the Hessian of V at ϕ_0, is the mass matrix of the fluctuating scalar field. (Compare with the lagrangian (3.18) for the free scalar field.) The kernel of the Hessian is comprised of massless fields, and if we are at a nondegenerate point of $\mathcal{M}_{\text{vac}}$ it is equal to the tangent space $T_{\phi_0}\mathcal{M}_{\text{vac}}$. The global symmetry group determines a subspace of $T_{\phi_0}\mathcal{M}_{\text{vac}}$—the tangent space to the orbit through ϕ_0. Massless scalars which lie in the tangent to the orbit are called *Goldstone bosons*. Here they are classical fields; a theorem in quantum field theory guarantees that they persist at the quantum level.

For gauge theory we fix a compact Lie group G and a bi-invariant scalar product $\langle \cdot, \cdot \rangle$ on the Lie algebra $\mathfrak{g}$. The field in the theory is a connection A on some principal G bundle. We identify connections which are isomorphic and only write gauge-invariant terms in the lagrangian. (The collection of all connections is a category, but the collection of equivalence classes under gauge transformations is a set.) The fundamental lagrangian in a pure gauge theory is

$$L = -\frac{1}{2}|F_A|^2\, |d^n x|, \tag{4.5}$$

where F_A is the curvature of the connection A. Here the bars denote both the norm in the Lie algebra and the norm on differential forms. The factor in front is chosen so that the kinetic term has the canonical normalization 1/2 the time derivative squared, as the reader may easily check in coordinates. The classical equation of motion is the *Yang-Mills equation*

$$d_A^* F_A = 0.$$

For any connection A we also have the Bianchi identity

$$d_A F_A = 0.$$

To write the energy density we split the curvature into the *electric field* 1-form E_A and the magnetic field 2-form B_A:

$$F_A = B_A - dx^0 \wedge E_A.$$

Here the electric and magnetic fields do not involve dx^0. Then the energy density is a generalization of a standard formula from classical electrodynamics:

$$\mathcal{E} = \left\{ \frac{1}{2}|E_A|^2 + \frac{1}{2}|B_A|^2 \right\} |d^{n-1}x|.$$

There is a single gauge equivalence class of minimal energy solutions—the trivial connections. In this case the moduli space of vacua is trivial:

$$\mathcal{M}_{\text{vac}} = \text{pt}\,. \tag{4.6}$$

We expand about the minimum, which simply amounts to replacing the connection A, which lives in an affine space, with a 1-form that lives in the associated vector space of translations. So the lagrangian is essentially unchanged and there are $\dim G$ massless 1-form fields.

Notice again that the vacuum solution is Poincaré-invariant. In fact, the equivalence class of trivial connections is the unique field configuration which is Poincaré-invariant. Though we won't discuss it here, the only minimal energy solution for a spinor field ψ with the usual kinetic term is $\psi \equiv 0$; it is also Poincaré-invariant.

Finally, we consider the most general bosonic field theory with scalar fields and gauge fields. It is defined by the following data:

(4.7)	G	compact Lie group
	$\langle\cdot,\cdot\rangle$	bi-invariant scalar product on $\mathfrak{g}$
	X	Riemannian manifold on which G acts by isometries
	$V\colon X \longrightarrow \mathbb{R}$	G-invariant function

There are two fields in the model:

(4.8)	A	connection on some principal G bundle $P \longrightarrow \check{M}^n$
	ϕ	section of the associated bundle $P \times_G X \longrightarrow \check{M}^n$

We can view ϕ as an equivariant map $\phi\colon P \to X$. Of course, a given example might be written in terms of many fields, but we can always consolidate them into the concise form written here. Then the fundamental lagrangian is

$$L = \left\{ -\frac{1}{2}|F_A|^2 + \frac{1}{2}|d_A\phi|^2 - \phi^* V \right\} |d^n x|. \tag{4.9}$$

Again we assume that the minimum of the potential V occurs at 0. Then the moduli space of vacua is

$$\mathcal{M}_{\text{vac}} = V^{-1}(0)/G. \tag{4.10}$$

The quotient by G comes from the gauge symmetry: a vacuum solution consists of a trivial connection A and a (covariant) constant scalar field ϕ. The global gauge transformations fix A and act on the constant scalar ϕ to produce (4.10). (We identify field configurations which differ by a gauge transformation.)

The global symmetry group H of the model is the group of isometries of X which commute with G and preserve V.

As before, we can do perturbation theory about some vacuum $[\phi_0] \in \mathcal{M}_{\text{vac}}$. We use the brackets to indicate the equivalence class of $\phi_0 \in V^{-1}(0)$ under the G action. Then we have

$T_{[\phi_0]}\mathcal{M}_{\text{vac}}$	massless scalar fields (if $[\phi_0]$ is nondegenerate)
$\text{Stab}_G\, \phi_0 \subset G$	massless gauge fields
$\text{Stab}_H[\phi_0] \subset H$	unbroken global symmetry group

The massless scalar fields at a singular point lie in the kernel of the Hessian of V on the quotient of $T_{\phi_0}X$ by the G action. Physicists sometimes say that the gauge group G is *broken* to the stabilizer subgroup $\text{Stab}_G\, \phi_0$. That reduction is the *Higgs mechanism.* To see how it works, let A_0 be a trivial connection and write $A = A_0 + \alpha$, where α is a 1-form with values in $\mathfrak{g} = \text{Lie}(G)$. Let $\{\zeta_i\}_{i-1}^k$ be a basis of $\mathfrak{g}$, with $\{\zeta_i\}_{i=1}^\ell$ a basis of $\text{Stab}_G\, \phi_0$. Write $\alpha = \alpha^i\zeta_i$. Now ζ_i induces a vector field $\hat{\zeta}_i$ on X. Then the quadratic terms in α in the expansion of (4.9) about $\phi = \phi_0$, $A = A_0$ are

$$-\frac{1}{2}|d\alpha|^2 + \sum_{i-1}^{k} \frac{1}{2}|\hat{\zeta}_i(\phi_0)|^2|\alpha^i|^2 = \sum_{i=1}^{\ell} -\frac{1}{2}|d\alpha^i|^2 + \sum_{j=\ell+1}^{k} -\frac{1}{2}|d\alpha^j|^2 + \frac{1}{2}|\hat{\zeta}_j(\phi_0)|^2|\alpha^j|^2.$$

Thus α^i $(i = 1, \dots, \ell)$ are massless 1-forms and α^j $(j = \ell+1, \dots, k)$ are massive 1-forms with masses $|\hat{\zeta}_j(\phi_0)|$.

This classical picture is usually corrected in the quantum theory. In a supersymmetric theory the corrections are constrained by the supersymmetry. An exciting development over the past few years is the realization that in many cases the constraints are stringent enough to determine the exact quantum picture of the moduli space, massless spectrum, and symmetry structure.

The superspacetime $M^{3|2}$

Recall from Lecture 3 that $M^{3|2}$ is defined from a 3-dimensional real vector space V with Lorentz metric and a 2-dimensional real vector space S^* which is a minimal real representation of the spin group $\text{Spin}(1,2) \cong SL(2,\mathbb{R})$. We fix isomorphisms $V \cong \text{Sym}^2 S^*$ and

$$\epsilon\colon \textstyle\bigwedge^2 S^* \to \mathbb{R}. \tag{4.11}$$

As coordinates on the superspace we fix

$$\begin{aligned} &\theta^1, \theta^2 && \text{on } \Pi S^*, \\ &y^{11}, y^{12}, y^{22} && \text{on } V. \end{aligned}$$

The indices on the y are symmetric and the underlying linear basis of V is determined from the underlying linear basis of S^* using the isomorphism $V \cong \operatorname{Sym}^2 S^*$. The y coordinates are related to the standard x coordinate system (4.1) by the formulas

$$\begin{aligned} y^{11} &= \frac{x^0 + x^1}{2}, \\ y^{22} &= \frac{x^0 - x^1}{2}; \\ y^{12} &= \frac{x^2}{2}. \end{aligned} \tag{4.12}$$

Instead of computing in coordinates, we use the global framing by the bi-invariant vector fields

$$\partial_{ab} = \frac{\partial}{\partial y^{ab}}$$

and the left invariant vector fields

$$D_a = \frac{\partial}{\partial \theta^a} - \theta^b \partial_{ab}.$$

These satisfy the bracket relation

$$[D_a, D_b] = -2\partial_{ab}.$$

Recall that we also use the right invariant vector fields

$$\tau_{Q_a} = \frac{\partial}{\partial \theta^a} + \theta^b \partial_{ab};$$

they commute with the D_a.

Our choice of coordinates defines an inclusion

$$i\colon \check{M}^3 \hookrightarrow M^{3|2}.$$

The pullback i^* simply sets $\theta^1 = \theta^2 = 0$.

Finally, we choose the basis of S^* to put the bilinear form (4.11) and its inverse into standard form:

$$\epsilon^{12} = \epsilon_{12} = 1. \tag{4.13}$$

Since ϵ is a *skew* form, there is a sign ambiguity in raising and lowering a single index. Rather than fix some convention, which we might easily forget, we agree to raise and lower only two indices at a time. This is unambiguous if we agree to raise each of the indices using the same sign convention. For example, the bilinear form ϵ canonically determines a bilinear form on the dual space, as indicated in (4.13). We warn the reader that physicists often use a different convention in which the form used on the dual space is not the naturally induced one. *Oy!*

Supersymmetric σ-model in 3 dimensions

Fix a Riemannian manifold X. We first construct the field theory directly on the superspacetime $M^{3|2}$. The field Φ is a scalar field on superspacetime with values in X:

$$\Phi\colon M^{3|2} \longrightarrow X.$$

Fields on superspacetime are usually called *superfields*. The (left) action of the super Poincaré group $P^{3|2}$ on $M^{3|2}$ induces an action on superfields by pullback. Recall that the infinitesimal action is by *right* invariant vector fields.

Just as we did for the superparticle (see (2.31)) we define *component fields* on Minkowski spacetime $\check{M}^3$ by restricting normal derivatives of the superfield to $\check{M}^3 \subset M^{3|2}$. Since the θ^a are nilpotent, a small number of normal derivatives determines the superfield completely. In other words, the Taylor expansion normal to $\check{M}^3$ is a finite power series in θ^a, and the component fields are essentially the coefficients in that expansion. However, rather than make such an expansion in some coordinate system, we use the global left-invariant framing to define global component fields. That said, set

$$\begin{aligned} \phi &= i^*\Phi \\ \psi_a &= i^* D_a \Phi \\ F &= -\frac{1}{2}\epsilon^{ab} i^* D_a D_b \Phi \end{aligned} \tag{4.14}$$

A few remarks are in order:

- The leading bosonic component ϕ is a scalar field on $\check{M}^3$ with values in X, namely

  $$\phi\colon \check{M}^3 \to X.$$

  The field ψ_a is odd, since the vector field D_a is odd. It is an odd section of the pullback tangent bundle. Together the two fields ψ_1, ψ_2 transform as a spinor on Minkowski spacetime, so we can combine them into

  $$\psi \in C^\infty\big(\check{M}^3, \phi^*\Pi(TX \otimes S)\big). \tag{4.15}$$

  The field F is even and is also a section of the pullback tangent bundle:

  $$F \in C^\infty(\check{M}^3, \phi^* TX).$$

- The outer (leftmost) derivative in the definition of F is a covariant derivative using the Levi-Civita connection, and in subsequent computations we will meet multiple covariant derivatives of Φ. We will use the fact that Levi-Civita is torsion-free when commuting the first two derivatives acting on Φ (see (2.3)); when commuting further derivatives we pick up curvature terms.
- In case $X = \mathbb{R}$, the fields ϕ, ψ comprise precisely the field content of a massless or massive scalar multiplet in $n = 3$ dimensions. (Recall the discussion of representations of $P^{3|2}$ in Lecture 3.) That is, if we write the free massless or massive lagrangian for these fields, then upon quantization the one-particle Hilbert space is that irreducible supersymmetry multiplet. So we are led to inquire: What is F doing here? We will see that F is an *auxiliary field* in the sense that it does not contribute physical degrees of freedom

in the quantization. In fact, F enters the lagrangian only algebraically—no derivatives—so the equations of motion determine F algebraically in terms of the other fields. We will eliminate F using this algebraic equation.

- Still, the presence of F allows us to write down supersymmetry transformation laws (see (4.17) below) for the component fields which close *off-shell*. If we substitute for F in the transformation laws, then this is no longer true.
- The component fields $\{\phi, \psi, F\}$ are often referred to as a *multiplet*, in this case a *scalar multiplet*.
- We can do the classical field theory in superspacetime with the superfield Φ or in ordinary spacetime with the component fields ϕ, ψ, F. In the first case we do calculus on $\mathcal{F}_\Phi \times M^{3|2}$; in the second calculus on $\mathcal{F}_{\{\phi,\psi,F\}} \times \check{M}^3$. The supersymmetry is *manifest* in the superfield formulation but not in the component formulation.

Next, we write down the supersymmetry transformation law for the component fields. Recall that in superspacetime the infinitesimal supersymmetry is the induced action on Φ of $-\tau_{Q_a}$, where we use a minus sign to obtain a left action as explained at the end of Lecture 2. Introduce odd parameters η^a and consider the action of the even vector field $-\eta^a \tau_{Q_a}$. This induces a vector field $\hat{\zeta}$ on the space of component fields $\mathcal{F}_{\{a,\psi,F\}}$. Quite generally, a component field f (see (4.14)) is defined by a formula

$$f = i^* D^r \Phi,$$

where 'D^r' denotes a sum of products of D_a. Then a short argument shows that under an infinitesimal supersymmetry transformation f transforms as

$$\hat{\zeta} f = -\eta^a i^* D_a D^r \Phi. \tag{4.16}$$

The right hand side can be rewritten in terms of component fields. This formula works in any model with a superfield formulation, not just in the $3|2$ supersymmetric σ-model. In the case at hand we find

$$\begin{aligned} \hat{\zeta}\phi &= -\eta^a \psi_a \\ \hat{\zeta}\psi_a &= \eta^b(\partial_{ab}\phi - \epsilon_{ab}F) \\ \hat{\zeta}F &= \eta^a\big[(\not{D}\psi)_a + \frac{1}{3}\epsilon^{bc}R(\psi_a, \psi_b)\psi_c\big], \end{aligned} \tag{4.17}$$

where R is the Riemann curvature tensor of X and the Dirac operator is

$$(\not{D}\psi)_a = -\epsilon^{bc}\partial_{ab}\psi_c.$$

This computation shows off the advantages of the superspacetime formulation—the supersymmetry transformation laws of the component fields are determined *a priori*.

The lagrangian density for the supersymmetric σ-model is

$$\mathcal{L} = |d^3x|\, d^2\theta\, \frac{1}{4}\epsilon^{ab}\langle D_a\Phi, D_b\Phi\rangle. \tag{4.18}$$

We have not given a detailed discussion of densities and integration on supermanifolds, and as before we finesse that point. Suffice it to say that $|d^3x|\, d^2\theta$ is a

$P^{3|2}$-invariant density on $M^{3|2}$. The lagrangian function ℓ which appears after it in the lagrangian density (4.18) is obviously $P^{3|2}$-invariant, since $P^{3|2}$ acts on the left and the D_a are left-invariant. So $\mathcal{L}$ is manifestly supersymmetric—invariant under $P^{3|2}$.

We need to "integrate out the θs" to define a component lagrangian L. Quite generally, for a superspacetime model on $M^{n|s}$ the superspacetime lagrangian density is

$$\mathcal{L} = |d^n x|\, d^s\theta\, \ell.$$

There is a standard notion of (Berezin) integration to integrate out the odd variables. Let

$$\pi\colon M^{n|s} \longrightarrow \check{M}^n$$

be the projection given by our construction of superspacetime (see (3.10)) and π_* the corresponding integration. Rather than integrate, our approach is to find in each case a certain sum of products of D_a, which we denote 'D^s', such that

$$\pi_*\mathcal{L} = (i^* D^s \ell + \Delta i^* \ell)\, |d^n x| \tag{4.19}$$

for some Poincaré invariant differential operator Δ on $\check{M}^n$. Then instead of using $\pi_*\mathcal{L}$ as the component lagrangian, we define the component lagrangian to be

$$L = (i^* D^s \ell)\, |d^n x|. \tag{4.20}$$

A few brief remarks:

- In general this differs from the straight integration of the odd variables, as indicated by the presence of Δ. The particular D^s that we use is chosen so that the component lagrangian density L involves only first derivatives of the fields.
- In this lecture we have $\Delta = 0$. Examples with $\Delta \neq 0$ occur in superspacetimes with $s = 4$ odd directions, as we will see in Lecture 5.
- Formula (4.20) is effective in that we find strings of D_a acting on superfields which we then express in terms of component fields using the commutation relations.
- The component lagrangian is supersymmetric, but the supersymmetry is not manifest. There is an explicit formula to determine the exact term by which it changes under a supersymmetry transformation. Recall from (2.21) that this term appears in the Noether current for the supersymmetry transformation, often called the *supercurrent*.

For $M^{3|2}$ we define the component lagrangian to be

$$L = i^*\big(-\frac{1}{2}\epsilon^{ab} D_a D_b\big)\ell\, |d^3 x|. \tag{4.21}$$

As remarked above, this agrees with the definition using the Berezinian integral $\int d^2\theta$. For the σ-model lagrangian (4.18) we have

$$L = -\frac{1}{8}\epsilon^{ab}\epsilon^{cd} i^* D_a D_b \langle D_c\Phi, D_d\Phi\rangle. \tag{4.22}$$

To illustrate the manipulations, consider a typical term in (4.22) without the numerical factor:

$$i^*\langle D_1 D_2 D_1 \Phi, D_2 \Phi\rangle.$$

We emphasize that the outer two derivatives are *covariant* derivatives. When we commute them we pick up a curvature term:

$$\langle D_1 D_2 D_1 \Phi, D_2 \Phi\rangle = -\langle D_2 D_1 D_1 \Phi, D_2 \Phi\rangle - 2\langle \partial_{12} D_1 \Phi, D_2 \Phi\rangle \\ + \langle R(D_1\Phi, D_2\Phi) D_1 \Phi, D_2 \Phi\rangle.$$

Since $D_1^2 = -\partial_{11}$ and $[\partial_{11}, D_2] = 0$, we have

$$D_2 D_1 D_1 \Phi = -D_2 \partial_{11} \Phi = -\partial_{11} D_2 \Phi.$$

Restricting to $\check{M}^3$ we obtain

$$i^*\langle D_1 D_2 D_1 \Phi, D_2 \Phi\rangle = \langle \partial_{11}\psi_2, \psi_2\rangle - 2\langle \partial_{12}\psi_1, \psi_2\rangle + \langle R(\psi_1, \psi_2)\psi_1, \psi_2\rangle.$$

Of course, there are systematic algebraic shortcuts one can develop for these manipulations. In any case, at the end of the day one obtains the component lagrangian for the supersymmetric σ-model:

$$L = \left\{ \frac{1}{2}|d\phi|^2 + \frac{1}{2}\langle \psi \not{D}_\phi \psi\rangle + \frac{1}{12}\epsilon^{ab}\epsilon^{cd}\langle \psi_a, R(\psi_b, \psi_c)\psi_d\rangle + \frac{1}{2}|F|^2 \right\} |d^3x|. \tag{4.23}$$

Some remarks:

- As promised, F appears only algebraically in the lagrangian. Its equation of motion is simply
$$F = 0.$$
- The spinor field ψ depends on ϕ (see (4.15)), as does the Dirac form. Also, the Dirac form is defined using a covariant derivative.
- The *bosonic lagrangian* is obtained by setting the fermion ψ to zero:
$$L_{\text{bos}} = \frac{1}{2}|d\phi|^2 \, |d^3x|$$
after eliminating the auxiliary field F. This is the nonlinear σ-model lagrangian (4.2) with zero potential. So we have indeed obtained a supersymmetric extension of the nonlinear σ-model with two supersymmetries in three dimensions.
- There is not much to say about this model: the moduli space of vacua is X and the global symmetry group is the group of isometries of X.

We need the formula for the *supercurrent* j_a, which is *minus* the Noether current corresponding to the supersymmetry τ_{Q_a}. Here we write the full $|-1|$-form on $\check{M}^3$:

$$j_a = \iota(\partial_{cd})\, |d^3x| \left\{ \epsilon^{cb}\epsilon^{de}\langle \partial_{ae}\phi, \psi_b\rangle - \epsilon^{bd}\delta_a^c \langle F, \psi_b\rangle \right\}.$$

We are only interested in this on-shell, in which case we could set $F = 0$, but we need the formula later in cases where $F \neq 0$. The *supercharges* are obtained by integrating j_a over a fixed time slice:

$$Q_1 = \int_{x^0=\text{const}} \Big\{ \langle \partial_{11}\phi, \psi_1 \rangle + \langle \partial_{12}\phi, \psi_2 \rangle \Big\} \, |dx^1 dx^2|$$
$$Q_2 = \int_{x^0=\text{const}} \Big\{ \langle \partial_{22}\phi, \psi_2 \rangle + \langle \partial_{12}\phi, \psi_1 \rangle \Big\} \, |dx^1 dx^2|$$

In these last formulas we set $F = 0$.

A supersymmetric potential

The theory is more fun when we add a potential term, which of course we must do in a supersymmetric way. This is easy in superspacetime. Let

$$h \colon X \longrightarrow \mathbb{R}$$

be a real-valued function on X. It is *not* the potential function V of the model; we compute V in terms of h below. Simply add the pullback of h by the superfield Φ to the superspacetime lagrangian (4.18):

$$|d^3x| \, d^2\theta \, \mathcal{L}' = \Big\{ \frac{1}{4} \epsilon^{ab} \langle D_a\Phi, D_b\Phi \rangle + \Phi^*(h) \Big\}.$$

It is easy to work out the contribution of the new term to the component lagrangian:

$$\Big\{ \langle F, \phi^* \operatorname{grad} h \rangle + \frac{1}{2} \epsilon^{ab} \phi^* (\operatorname{Hess} h)(\psi_a, \psi_b) \Big\} \, |d^3x|. \tag{4.24}$$

So the new component lagrangian is the sum of (4.23) and (4.24). Now the equation of motion of F is more interesting than before:

$$F = -\phi^* \operatorname{grad} h.$$

So upon eliminating F from the component lagrangian we obtain

$$\begin{aligned} L' = \Big\{ \frac{1}{2}|d\phi|^2 + \frac{1}{2}\langle \psi \not{D}_\phi \psi \rangle - \frac{1}{2}\phi^* |\operatorname{grad} h|^2 + \frac{1}{2}\epsilon^{ab}\phi^*(\operatorname{Hess} h)(\psi_a, \psi_b) \\ + \frac{1}{12}\epsilon^{ab}\epsilon^{cd}\langle \psi_a, R(\psi_b, \psi_c)\psi_d \rangle \Big\} \, |d^3x|. \end{aligned} \tag{4.25}$$

The presence of h leads to two new terms:

- A potential energy
$$V = \frac{1}{2} |\operatorname{grad} h|^2$$
for the scalar field ϕ. Note that the potential is nonnegative, as it must be in a supersymmetric theory.
- A mass term
$$\frac{1}{2}\epsilon^{ab}\phi^*(\operatorname{Hess} h)(\psi_a, \psi_b)$$
for the fermions.

The bosonic lagrangian in this case is

$$L_{\text{bos}} = \left\{ \frac{1}{2}|d\phi|^2 - \frac{1}{2}\phi^* |\operatorname{grad} h|^2 \right\} |d^3x|,$$

a nonlinear σ-model with nonzero potential.

We also record the *on-shell* supersymmetry transformation laws for the physical fields ϕ, ψ from (4.17), using the equation of motion for F:

$$\begin{aligned} \hat{\zeta}\phi &= -\eta^a \psi_a \\ \hat{\zeta}\psi_a &= \eta^b(\partial_{ab}\phi + \epsilon_{ab}\phi^* \operatorname{grad} h). \end{aligned} \tag{4.26}$$

The supercharges are now given by the formulas

$$\begin{aligned} Q_1 &= \int_{x^0=\text{const}} \left\{ \langle \partial_{11}\phi, \psi_1 \rangle + \langle \partial_{12}\phi, \psi_2 \rangle - \langle \phi^* \operatorname{grad} h, \psi_2 \rangle \right\} |dx^1 dx^2| \\ Q_2 &= \int_{x^0=\text{const}} \left\{ \langle \partial_{22}\phi, \psi_2 \rangle + \langle \partial_{12}\phi, \psi_1 \rangle + \langle \phi^* \operatorname{grad} h, \psi_1 \rangle \right\} |dx^1 dx^2| \end{aligned} \tag{4.27}$$

Now we investigate the classical vacuum states—the minimal energy field configurations. As we mentioned before these occur when $\psi = 0$ and $\phi \equiv \phi_0$ is constant. If ϕ_0 is at a minimum of the potential V—in other words, $V(\phi_0) = 0$—then ϕ_0 is a critical point of h. These field configurations are invariant under supersymmetry. In other words, the vector field $\hat{\zeta}$ vanishes at such points of field space, as is obvious from (4.26). Thus we say that *supersymmetry is unbroken* in these vacua. It is natural to assume that h is a Morse function, or perhaps a generalized Morse function in the sense of Bott. Then we might have critical manifolds, and at vacua which lie on such manifolds there are massless scalars and massless fermions. The fact that supersymmetry is unbroken means that these come in supersymmetric pairs, i.e., in multiplets. More generally, the masses of the scalar fields and of the fermions agree at such a vacuum. This is easily verified by computing the masses directly from the lagrangian (4.25). (Recall how masses appear in the free lagrangian (3.18) for the scalar field and (3.23) for the spinor field.) For the scalar field ϕ, the squares of the masses are the eigenvalues of the Hessian of the potential V, which at a critical point of h are the squares of the eigenvalues of $\operatorname{Hess} h$. On the other hand, the mass matrix M for the fermions is $\operatorname{Hess} h$, and so the fermion mass squares are again the squares of the eigenvalues of the Hessian.

We can also consider a vacuum which is a local minimum of V, but not a global minimum. (A critical point of V which is not a local minimum is unstable.) For such a local minimum ϕ_0 we need to shift the energy by $-V(\phi_0)$ in order to have zero energy for the vacuum. Since

$$dV = \langle \operatorname{grad} h, \nabla \operatorname{grad} h \rangle,$$

such critical points of V occur when $\operatorname{grad} h$ is nonzero but in the kernel of $\operatorname{Hess} h$. Such a point could be isolated, and at such a vacuum we might have no massless scalar fields. But there is a massless fermion since the fermion mass matrix $\operatorname{Hess} h$ has a kernel. In fact, this massless fermion can be predicted from the supersymmetry. Notice that this vacuum is *not* supersymmetric—the vector field $\hat{\zeta}$ in (4.26)

does not vanish there. So we say that *supersymmetry is broken* in this vacuum. The Poincaré symmetry is always assumed to be unbroken in a vacuum, but as we see this is not true for the *super* Poincaré symmetry. Now just as broken (even) global symmetries lead to massless bosons—Goldstone bosons—so too do broken odd symmetries lead to massless fermions: *Goldstone fermions.*

Dimensional reduction to $n = 2$ dimensions

We dimensionally reduce the model in components, though we could dimensionally reduce the superspacetime formulation to obtain a superspacetime formulation in $n = 2$ dimensions.

Let's begin by collecting the formulas we need. Restrict to fields f which satisfy $\partial_{12} f = 0$. (Recall from (4.12) that y^{12} is a spatial direction.) It is convenient to use *lightcone coordinates* on $\check{M}^2$, or equivalently to use a basis of vector fields ∂_+, ∂_- which are lightlike. (See (3.14).) Recall that these are related to ∂_0, ∂_1 by

$$\begin{aligned} \partial_+ + \partial_- &= 2\partial_0 \\ \partial_+ - \partial_- &= 2\partial_1 . \end{aligned}$$

Also, for spinors we use '+' for the index '1' and '−' for the index '2'. (In more invariant language, we decompose spinors under the inclusion $\mathrm{Spin}(1,1) \subset \mathrm{Spin}(1,2)$.) Then from (4.26) we find the supersymmetry transformations

$$\begin{aligned} \hat{\zeta}\phi &= -(\eta^+\psi_+ + \eta^-\psi\ ) \\ \hat{\zeta}\psi_+ &= \eta^+\partial_+\phi + \eta^-\phi^* \operatorname{grad} h \\ \hat{\zeta}\psi_- &= \eta^-\partial\ \ \phi - \eta^+\phi^* \operatorname{grad} h \end{aligned} \tag{4.28}$$

and from (4.27) the supercharges

$$\begin{aligned} Q_+ &= \int_{x^0=\mathrm{const}} \Big\{ \langle \partial_+\phi, \psi_+\rangle - \langle \phi^* \operatorname{grad} h, \psi_-\rangle \Big\} \, |dx^1| \\ Q_- &= \int_{x^0=\mathrm{const}} \Big\{ \langle \partial_-\phi, \psi_-\rangle + \langle \phi^* \operatorname{grad} h, \psi_+\rangle \Big\} \, |dx^1| . \end{aligned} \tag{4.29}$$

Field theory in $n = 2$ dimensions is special for many reasons, among them the fact that spatial infinity is disconnected. This has the following consequence. Let $\mathcal{FE}_{\mathrm{bos}}$ denote the space of bosonic field configurations on space of finite energy. Now the energy density of a field $\phi(x^1)$ on space is, according to (4.4),

$$\begin{aligned} \mathcal{E} &= \Big\{ \frac{1}{2}\,|\partial_1\phi|^2 + \frac{1}{2}\,\phi^* |\operatorname{grad} h|^2 \Big\} \, |dx^1| \\ &= \frac{1}{2}\,|\partial_1\phi \pm \phi^* \operatorname{grad} h|^2 \, |dx^1| \ \mp\ d\phi^*(h). \end{aligned} \tag{4.30}$$

If a field $\phi(x^1)$ on space has finite energy, then the integral of the first term is finite, in which case $\phi(x^1)$ has limits as $x^1 \to \pm\infty$ which are critical points of h. The integral over space of the topological term is plus or minus the difference

$$Z = h\big(\phi(\infty)\big) - h\big(\phi(-\infty)\big)$$

of the limiting values. So the space $\mathcal{FE}_{\text{bos}}$ of finite energy field configurations splits into a disjoint union according to the limiting values at plus or minus infinity. The parameter space for the components is the Cartesian square $\pi_0 \operatorname{Crit}(h)^{\times 2}$ of the set of components of the critical point set. The *central charge* Z is constant on each component.

Note that in $n = 3$ spacetime dimensions spatial infinity is connected and a field with finite energy has a single limit at infinity. So the components of $\mathcal{FE}_{\text{bos}}$ in that case are labeled by $\pi_0 \operatorname{Crit}(h)$.

Example. A typical example is $X = \mathbb{R}$ with

$$h(\phi) = \frac{\phi^3}{3} - a^2\phi, \qquad a \in \mathbb{R}.$$

Then we have a quartic potential

$$V(\phi) = \frac{1}{2}(\phi^2 - a^2)^2 \tag{4.31}$$

with nondegenerate zeros at $\phi = \pm a^2$. So in the $n = 2$-dimensional theory $\mathcal{FE}_{\text{bos}}$ has four components.

In a "diagonal" component of $\mathcal{FE}_{\text{bos}}$, where $h(\infty) = h(-\infty)$, there are field configurations of zero energy: constant fields with values in the critical point set. (For an isolated critical point there is a unique such field.) These are the classical vacua. But in components where $h(\infty) \neq h(-\infty)$ there are no vacua. In any component the energy E, which is the spatial integral of the energy density $\mathcal{E}$ (4.30), is bounded below by the absolute value of the central charge:

$$E \geq |Z|. \tag{4.32}$$

From (4.30) we see that a field configuration of minimal energy satisfies the *first order* differential equation for a flow line:

$$\partial_1\phi \pm \phi^* \operatorname{grad} h = 0. \tag{4.33}$$

Equations of the form (4.30) occur in certain calculus of variations problems: a functional is written as a square plus a topological term, and minima of the functional solve a first order differential equation (whereas the Euler-Lagrange equations are second order). Now we have an interpretation of this argument in terms of supersymmetric field theory.

In general, nonvacuum field configurations of locally minimal energy are called *solitons*. In the example (4.31) they are sometimes called *kinks* or *lumps*. We can take the soliton to be constant in time. Physically it represents a stable localized lump sitting still in space. Upon quantization we might expect that it gives a particle, and because of the inequality (4.32) we can expect it to be massive. In more detail: When we quantize the theory we need to quantize the space $\mathcal{M}$ of all finite energy classical solutions to the field equations. This space divides into components as indicated above. The lowest energy state in a diagonal component is a vacuum of zero energy. The small fluctuations around the vacuum give massless and massive particles according to the Hessian of V, at least in perturbation theory.

Nondiagonal components have no state of zero mass. Rather, the states of smallest mass saturate the inequality (4.32) (where we replace energy by mass) and form a symplectic manifold on which the Poincaré group acts. The quantization of this symplectic manifold is then a collection of *quantum solitons*. They are massive, and in perturbation theory their mass is very large compared to the masses of the particles constructed from small fluctuations.

Returning to the Morse theory example, in the simplest case there is a single geometric flow line between distinct critical points. However, spatial translations act freely on the space of parametrized flow lines $\phi(x^1)$, and the space of static solitons is one-dimensional. We act on these flow lines by Lorentz boosts to obtain moving solitons; the collection of these is two-dimensional and is a homogeneous space for the Poincaré group. We can identify this space with the tangent bundle to the space of parametrized flow lines, and upon quantization we obtain a massive scalar particle.

Our discussion so far has ignored the fermion and the supersymmetry. In fact, the central charge Z appears in the supersymmetry algebra as was anticipated in (3.14), where we discussed a central extension of the supersymmetry algebra $\mathfrak{p}^{2|(1,1)}$. Now we realize the central extension classically by computing the Poisson bracket $\{Q_+, Q_-\}$ of the supersymmetry charges. The Poisson bracket of Noether charges can be computed in different ways (see (2.18)), and is in fact the spatial integral of a Poisson bracket of Noether currents. We use the explicit formulas (4.29) and note that the nontrivial contributions come from the brackets $\{\psi_+, \psi_+\}$ and $\{\psi_-, \psi_-\}$. The result is

$$\begin{aligned}
\frac{1}{2}\{Q_+, Q_-\} &= \frac{1}{2}\int_{x^0=\text{const}} \Big\{\langle \partial_+\phi, \phi^* \operatorname{grad} h\rangle - \langle \partial_-\phi, \phi^* \operatorname{grad} h\rangle\Big\}\, |dx^1| \\
&= \int_{x^0=\text{const}} \langle \partial_1\phi, \phi^* \operatorname{grad} h\rangle\, |dx^1| \\
&= \int_{x^0=\text{const}} \partial_1(\phi^* h)\, |dx^1| \\
&= Z.
\end{aligned}$$

So we obtain a central extension of the supersymmetry algebra, as promised. The energy inequality (4.32) is the classical version of (3.17), and it follows from the supersymmetry algebra as in (3.16). Field configurations which saturate this classical *BPS inequality* are annihilated by the supersymmetry $\tau_{Q_+} + \tau_{Q_-}$ or by $\tau_{Q_+} - \tau_{Q_-}$. This provides an alternative derivation—using supersymmetry—of the first order equation (4.33) for minimal energy bosonic field configurations. Namely, we look at the supersymmetry transformation (4.28) and ask that the vector field $\hat{\zeta}$ vanish. Inspecting the variation of the fermions ψ_+, ψ_- we learn that such *BPS field configurations* satisfy

$$\begin{aligned}
\partial_+\phi \pm \phi^* \operatorname{grad} h &= 0 \\
\pm\partial_-\phi - \phi^* \operatorname{grad} h &= 0.
\end{aligned} \tag{4.34}$$

These equations are equivalent to the equations

$$\begin{aligned}
\partial_0\phi &= 0 \\
\partial_1\phi \mp \phi^* \operatorname{grad} h &= 0,
\end{aligned}$$

which are the equations for a static flow line. We also obtain an equation on fermions by requiring that the variation of the boson ϕ vanish in (4.28):

$$\psi_+ \pm \psi \quad = 0. \tag{4.35}$$

To analyze this we must also bring in the equations of motion for the fermions, which are

$$\begin{aligned} -\partial_+\psi_- &= R(\psi_+,\psi_-)\psi_+ + \phi^*(\nabla \operatorname{grad} h)(\psi_+) \\ \partial_-\psi_+ &= -R(\psi_-,\psi_+)\psi_- + \phi^*(\nabla \operatorname{grad} h)(\psi_-). \end{aligned} \tag{4.36}$$

Combining (4.35) and (4.36) we see that the curvature terms vanish by the Bianchi identity and we are left with equations for a single fermion ψ_+:

$$\begin{aligned} \partial_0\psi_+ &= 0 \\ \partial_1\psi_+ \mp \phi^*(\nabla \operatorname{grad} h)(\psi_+) &= 0. \end{aligned} \tag{4.37}$$

Equations (4.37) are the variations of equations (4.34) in the direction ψ_+. Thus the BPS fermion field ψ_+ is an odd tangent vector to the manifold of flow lines.

Recapping: The BPS condition leads to a first order equation on bosons which implies the second order equation of motion. This demonstrates that classical fermionic fields, which are difficult to visualize in a geometric manner, can impact classical bosonic fields.

Finally, we comment on the quantization of these solitons including the fermions. In this supersymmetric theory we expect to get an action of the supersymmetry group, and indeed what we must quantize is now a symplectic *super*manifold. The bosonic part as before is the tangent bundle TN to a space of parametrized flow lines N. The fermion field, according to (4.37), is an odd tangent vector, so the supermanifold can be described as

$$\pi^*\Pi TN \longrightarrow TN,$$

where $\pi\colon TN \to N$. This is the same supermanifold which entered into our discussion of the superparticle—see (2.15). So we see that the classical system we must quantize is supersymmetric quantum mechanics on the space N of static solitons, here parametrized flow lines. In models where we start with more supersymmetry, we end up with supersymmetric quantum mechanics with more supersymmetry, from which we learn about the geometry of the space of solitons.

Dimensional reduction to $n = 1$

We make some informal remarks about the *quantum* mechanical model.

Classically, we reduce to a mechanical model by requiring the fields in the $n = 3$-dimensional model to be invariant under all spatial translations. We obtain a model with supersymmetry group $P^{1|2}$. It has a superspacetime formulation on $M^{1|2}$, which is the dimensional reduction of the superspacetime formulation in $n = 3$ dimensions. This theory has twice the minimal amount of supersymmetry. In particular, it has twice as much supersymmetry as the superparticle model considered in Lecture 2. There we indicated that the quantum Hilbert space of that model is the space of spinor fields on X. With twice as much supersymmetry we

obtain instead the space of differential forms on X as the quantum Hilbert space; the $\mathbb{Z}/2$-grading is by the parity of the degree. In the minimal superparticle there is a single supercharge, and its quantization is the Dirac operator. Here there are two supercharges (4.29) and the corresponding quantum operators are, up to factors, the first order differential operators $d + d^*$ and $\sqrt{-1}(d - d^*)$. This is if we have no potential term. In case there is a potential term, then the second term in the supercharges (4.29) gives an additional term which is a combination of exterior and interior multiplication by dh. In this way we obtain the modification of the de Rham complex used by Witten in his study of Morse theory. The hamiltonian in this model is the Hodge laplacian on differential forms modified by lower order terms involving dh. The solitons we discussed in $n = 2$ dimensions are *instantons* in $n = 1$ dimension, and they enter into the Morse theory discussion.

The dimensional reduction from $n = 3$ dimensions to $n = 1$ has a global $SO(2)$ symmetry from spatial rotations. From the $n = 3$-dimensional point of view these symmetries are part of the Poincaré group; from the $n = 1$-dimensional point of view they are an example of an R-symmetry. In the quantum mechanical theory this global $SO(2)$ symmetry gives a $\mathbb{Z}$-grading on the quantum Hilbert space; it is simply the usual $\mathbb{Z}$-grading on differential forms.

Gauge theory on $M^{3|2}$

Fix a compact Lie group G and a bi-invariant inner product $\langle \cdot, \cdot \rangle$ on its Lie algebra. Just as in the bosonic case, the field in the superspacetime formulation of a minimal supersymmetric pure gauge theory is a G-connection. However, in the supersymmetric case the connection $\mathcal{A}$ on superspacetime is *constrained* to be flat along the odd distribution spanned by the left invariant vector fields D_a:

$$\iota(D_a)\iota(D_b)F_{\mathcal{A}} = 0 \qquad \text{for all } a, b. \tag{4.38}$$

Here $F_{\mathcal{A}}$ is the curvature of $\mathcal{A}$. Recall from Table 5 that minimal supersymmetric gauge theories have a superspacetime formulation on $M^{2|(1,0)}$, $M^{3|2}$, $M^{4|4}$, and $M^{6|(8,0)}$.

We specialize to pure supersymmetric gauge theory on $M^{3|2}$. Rewrite the constraint (4.38) in terms of covariant derivatives $\mathcal{D}_a$ and ∇_{ab} along the vector fields D_a, ∂_{ab} (acting on the adjoint bundle):

$$[\mathcal{D}_a, \mathcal{D}_b] = -2\nabla_{ab}. \tag{4.39}$$

In this case it is easy to find an unconstrained field—a partial connection along the odd distribution. In other words, given any $\mathcal{D}_a$ we use (4.39) to define ∇_{ab} unambiguously, since $V \cong \operatorname{Sym}^2 S^*$.

The Bianchi identities together with the constraint (4.39) imply that all components of the curvature can be written in terms of

$$\Lambda_a = \frac{1}{3}\epsilon^{bc}[\nabla_{ab}, \mathcal{D}_c].$$

This is a particular component of the curvature, and it is a section of the adjoint bundle. We define the component fields corresponding to $\mathcal{A}$ to be

$$\begin{aligned} A &= i^*\mathcal{A}, && \text{connection on a bundle } P \to \check{M}^3; \\ \lambda_a &= i^*\Lambda_a, && \lambda \in C^\infty\big(\check{M}^3, \Pi(\operatorname{ad} P \otimes S)\big). \end{aligned} \tag{4.40}$$

There are no auxiliary fields in this model. The space of constrained connections $\mathcal{A}$ up to equivalence is isomorphic to the space of component fields $\{A, \lambda\}$ up to gauge equivalence. (In fact, the collection of fields in each case is a category, and the categories are equivalent.) Note that in case $G = \mathbb{T}$ is the circle group, then the component fields comprise a *vector multiplet* in $n = 3$ dimensions with $s = 2$ supersymmetries. That is, when we quantize the free theory, as in Lecture 3, we obtain the vector multiplet representation of $P^{3|2}$.

For the curvature of A we write

$$*F_A = f_{ab}\, dy^{ab}.$$

Then the supersymmetry transformation laws of the component fields, computed from the general formula (4.16), are

$$\begin{aligned} (\hat{\zeta}A)_{ab} &= -\eta^c(\epsilon_{ac}\lambda_b + \epsilon_{bc}\lambda_a) \\ \hat{\zeta}\lambda_a &= -\eta^b f_{ab}. \end{aligned} \tag{4.41}$$

The variation of the connection A is an odd 1-form with values in the adjoint bundle; the coefficient of dy^{ab} is written in (4.41).

The superfield lagrangian is

$$\mathcal{L} = |d^3x|\, d^2\theta \left\{ -\frac{1}{4}\epsilon^{ab}\langle \Lambda_a, \Lambda_b \rangle \right\}, \tag{4.42}$$

from which we derive the component lagrangian

$$L = \left\{ -\frac{1}{2}|F_A|^2 + \frac{1}{2}\langle \lambda \not{D}_A \lambda \rangle \right\} |d^3x|$$

using (4.21). Notice that the bosonic lagrangian (set $\lambda = 0$) is the standard Yang-Mills lagrangian (4.5), so this is indeed a minimal supersymmetric extension of bosonic Yang-Mills theory.

Just as for bosonic pure gauge theory, there is not much to say about the classical theory. For example, the moduli space of vacua is a point (4.6). The coupling of the supersymmetric gauge theory to the supersymmetric σ-model is straightforward in superspace; the construction of the theory is the exact analog of the bosonic construction in (4.7)–(4.9). There are no surprises. We will have more interesting stories to tell in Lecture 5 when we consider theories with more supersymmetries and their dimensional reductions.

LECTURE 5
Theories with More Supersymmetry

The title of this lecture suggests more than we can possibly deliver. It encompasses many possible theories: theories in $n \leq 4$ dimensions with $s = 4$ supersymmetries, theories in $n \leq 6$ dimensions with $s = 8$ supersymmetries, and theories in $n \leq 10$ dimensions with $s = 16$ supersymmetries. And perhaps it also suggests supergravity. Sorry! Our more modest goal is to introduce some of the basic cases and to discuss some of their most salient features. To that end we emphasize the superspacetime $M^{4\,4}$ and theories with $P^{4|4}$ symmetry. They and their dimensional reductions are also useful for discussing theories with more than 4 supersymmetries. Also, we summarize some of the geometry which enters into theories with 8 or 16 supersymmetries in the maximal dimension. As far as the dimensional reductions, each exhibits its own peculiarities and we select a few representative cases to illustrate some of the phenomena which occur. We will not discuss theories with *chiral* supersymmetry beyond the minimal amount.

We begin with bosonic theories again. First, we add some perspective to our general discussion at the beginning of Lecture 4 about classical bosonic lagrangians. Namely, in the application to *quantum* field theory we must distinguish between lagrangians which can be used to define *fundamental, microscopic* theories valid at arbitrarily small distance (large energy) and lagrangians used to describe an *effective, macroscopic* theory at large distance (small energy). The discussion is very terse and meant to help the reader understand a physicist's perspective; it is not necessary for what follows. There is an additional feature of bosonic theories we point out which we have not yet encountered: dimensional reduction of a gauge field produces additional scalar fields.

Then we turn to our main task, which is the description of theories with symmetry group $P^{4|4}$. We write these theories in the superspacetime $M^{4\,4}$. The description of the σ-model is similar to that of the σ-model in $M^{3|2}$ treated in Lecture 4, only now the superfield is constrained to be *chiral* and the target manifold must be *Kähler*. Also, the potential energy is built out of a *holomorphic* function called the *superpotential*. Gauge theory on $M^{4|4}$ has a similar description to that on $M^{3|2}$, but now the coupled model with scalars in a Kähler manifold X exhibits a new feature: a term in the potential energy coming from the *moment map* of the G action on X.

The target manifold of a σ-model with 8 supersymmetries must be *hyperkähler*. In the coupled model we meet the hyperkähler moment map, analogously to the Kähler case. As indicated in Table 5 there is no σ-model with 16 supersymmetries.

There is, however, a pure supersymmetric gauge theory in $n \leq 10$ dimensions. Neither the σ-model with 8 supersymmetries nor the gauge theory with 16 supersymmetries has a superspacetime description (off-shell). At the end of the lecture we demonstrate using components that the minimal supersymmetric Yang-Mills lagrangian is indeed supersymmetric in $n = 10$ dimensions.

We meet more interesting geometry when we consider dimensional reductions of these models, especially in the possible effective lagrangians. This is a large topic and we include only a few examples to illustrate some of the ideas involved. One notable feature which has no analog in ordinary Minkowski spacetime: There exist nontrivial super Poincaré invariant connections in superspacetimes. In some cases with extended supersymmetry we can use these as background connections to obtain special mass terms in lagrangians.

Fundamental vs. effective lagrangians

The bosonic lagrangian (4.9) we considered in Lecture 4 covers many cases of interest, but is hardly the most general classical lagrangian with a connection A and a scalar field ϕ. For example, we might write terms which involve higher powers of first derivatives—say, $|F_A|^4$ or $|d_A\phi|^4$—or terms involving higher derivatives—say $|\nabla_A F_A|^2$ or $|\nabla_A d_A \phi|^6$. Even if we stick to terms which are at most quadratic in the first derivatives, there are more general terms which we can write. For example, we might allow the coefficient in front of the Yang-Mills action to depend on the scalar field:

$$-\frac{1}{2g^2(\phi)}|F_A|^2\,|d^n x|, \tag{5.1}$$

where $g^2\colon X \to \mathbb{R}^{>0}$ is a G-invariant function. Also, there are special terms we can write in specific dimensions. In $n = 2$ dimensions if $\langle\cdot\rangle\colon \mathfrak{g} \to \mathbb{R}$ is an infinitesimal character of G—a trace on $\mathfrak{g}$—then we have the *topological term*

$$\theta\langle F_A\rangle. \tag{5.2}$$

Here θ is a real coupling constant. The term is called topological since it is locally constant on the space of fields with finite action. Notice that it is a 2-form, and to turn it into a density we need to fix an orientation. Said differently, theories with nonzero θ are not invariant under orientation-reversing transformations in the Lorentz group. Such terms are said to be *CP noninvariant*. Another 2-form term possible in 2-dimensional theories with a $\mathfrak{g}$-valued scalar field σ is

$$\langle F_A, \sigma\rangle. \tag{5.3}$$

Again the 2-form breaks CP invariance. (For $G = \mathbb{T}$ the circle group, (5.3) is obtained from (5.2) by promoting the real constant θ to a variable Lie($\mathbb{T}$)-valued field σ.) In $n = 4$ dimensions we use the inner product $\langle\cdot,\cdot\rangle$ on $\mathfrak{g}$ to write a topological term

$$\theta\langle F_A \wedge F_A\rangle \tag{5.4}$$

for θ a real coupling constant. In $n = 3$-dimensional gauge theories we can also include a Chern-Simons term.

As geometers we study many of these classical lagrangians on curved Riemannian manifolds M, depending on the geometric problem of interest. In *quantum* field theory lagrangians are used both to describe *fundamental* (*microscopic*) theories and *effective* (*macroscopic*) theories. It is often said that the microscopic theory is the *question* (*problem*) and the macroscopic theory the *answer* (*solution*). Given a well-posed question there is in principle a method of computing the answer, but a direct approach is usually intractable. One tries instead to use physical insight and symmetry principles to determine the form of the effective lagrangian.

A fundamental theory should make sense at arbitrarily high energies, and so microscopic lagrangians are constrained by *renormalizability* and *asymptotic freedom*. Both concepts refer to perturbation theory everything is computed as a perturbation of a free theory. And both put constraints on the data (4.7) used to define the theory. As we might expect, the constraints depend strongly on the dimension. In $n = 2$ dimensions a σ-model is renormalizable for any target manifold X, but in higher dimensions the theory is renormalizable only if X is a vector space. Then for renormalizability in $n = 3$ dimensions the potential V must be a polynomial of degree at most 6, and in $n = 4$ dimensions it must be a polynomial of degree at most 4. Gauge theories are renormalizable in $n \leq 4$ dimensions. Renormalizability excludes higher derivative terms and a nontrivial coupling (5.1), though the terms (5.2)–(5.4) described above are possible. There are no renormalizable (nonfree) theories above $n = 4$ dimensions. Asymptotic freedom is determined by computing the 1-loop β function. It also depends on the spinor fields in the theory, and since we have not included those we will not try to describe the precise statements. Suffice it to say that there are asymptotically free theories and that in $n = 4$ dimensions they involve nonabelian gauge fields.

An *effective lagrangian* is used to describe the behavior of a quantum field theory at large distances (low energies). Here the physical input goes beyond the purely formal. The first step is to determine what the proper degrees of freedom are to describe the long-range behavior of the system. These are massless fields, but they may be quite different than the fields which are used in the fundamental lagrangian to define the theory. For example, we may start with a gauge theory with gauge group G but at long distance only see a subgroup of G because of the Higgs mechanism. Or, say in $n = 3$ dimensions, we may start with a nonabelian gauge theory and then discover that at long distances we have a massless abelian gauge field. In that case we should write the effective lagrangian in terms of a dual scalar field; recall that we mentioned this duality at the end of Lecture 3. Sometimes in the effective action we use a scalar field which is made as the product of spinor fields. In other cases we find fields whose expression in terms of the fields in the fundamental lagrangian is highly nonlocal and nontrivial (and nonexplicit). In $n = 4$ dimensions we might find composite massless gauge fields with gauge group G' which is not a subgroup of the gauge group G in the fundamental theory. As should be clear, identifying the proper degrees of freedom at long distances requires physical insight into the *dynamics* of the theory.

Once the fields are determined, the effective action is usually said to include all possible terms compatible with the symmetries of the theory, but with coefficients to be determined. (When the symmetries of the theory include supersymmetries, these coefficients are often highly constrained and more can be said.) Terms in the effective lagrangian are organized into a *derivative expansion*, where the leading order contributions come from terms with a small number of derivatives. This

reflects the fact that we are studying low energy field configurations which are essentially constant. As in (4.9) one often restricts to terms at most quadratic in first derivatives, but now there is no renormalizability constraint. Nor is asymptotic freedom—a property which refers to the behavior at short distances—relevant. The corresponding property at large distances is *infrared freedom*, and this is necessary if we want to describe the long-range behavior as perturbation of a free theory. (It is not necessarily true that a theory is infrared free. It may instead be described by a nontrivial conformal field theory at long distance, often described by saying that the theory "flows to a nontrivial fixed point". There are many nontrivial conformal theories in $n = 2$ dimensions and they exist in higher dimensions as well.) Now if the effective action is written in terms of massless scalars only, and if there is no potential function V, then the manifold X on which the massless scalars take their values is called the *quantum moduli space*. It often differs from the classical moduli space and is one of the key objects of interest. The quantum moduli space is a *complete* Riemannian "manifold" possibly with singularities. At a singularity there is new physics—more fields are massless. We can include massive scalars and a potential, in which case the quantum moduli space is determined as in (4.10). For example, near a singularity in the quantum moduli space X there may be a description in terms of G-connections and scalar fields in a manifold Y with potential V such that the moduli space $V^{-1}(0)/G$ parametrizes X in a neighborhood of the singularity. This illustrates that there may be no single global description of the quantum moduli space—for example, different fields may be used to describe different local regions. As for massless gauge fields, in general the coupling constants in the effective lagrangian depend on the scalar fields as in (5.1). (In fact, there is a coupling constant for each simple or abelian factor of the gauge group G. Put differently, the invariant inner product $\langle\cdot,\cdot\rangle$ is a function of the scalar fields.)

Although we have only discussed bosonic lagrangians in this subsection, we can and should also include fermions in both microscopic and macroscopic lagrangians. Of course, they are required in supersymmetric lagrangians.

It should be said that without supersymmetry the massless scalars one encounters are usually Goldstone bosons. That is, they are due to spontaneous global symmetry breaking. In supersymmetric theories there are nontrivial quantum moduli spaces which are not homogeneous.

Finally, in applications to geometry we often analytically continue a lagrangian to Euclidean space and then try to formulate a version on curved Riemannian manifolds. Thus, for example, geometers and topologists get great mileage by studying the Yang-Mills equations on Riemannian 4-manifolds. As for supersymmetric lagrangians, they can be formulated on curved Riemannian manifolds in many cases using a device called *topological twisting*, but we will not consider them here.

Dimensional reduction of gauge fields

Suppose $\mathbf{A}$ is gauge field on Minkowski spacetime $\check{M}^{n+r}$ invariant under the spatial translations $\partial/\partial x^j$, $j = n, \dots, n+r-1$. (The indices $j, k, \dots$ in this subsection always run over the last r spatial directions. We use standard coordinates on Minkowski spacetime as in previous lectures.) This means that the indicated spatial translations have been lifted to commuting vector fields on the total space of the principal bundle $P \to \check{M}^{n+r}$ which hosts the connection $\mathbf{A}$, and that the Lie

derivative of the connection form along these vector fields vanishes. Then we write

$$\mathbf{A} = A + \sigma_j dx^j,$$

where A does not involve any dx^j. We view $\check{M}^n$ as the quotient of $\check{M}^{n+r}$ by the last r spatial translations. Then A is a connection on $\check{M}^n$ on the quotient bundle $\overline{P} \to \check{M}^n$ and each σ_j is a scalar field with values in the adjoint bundle $\operatorname{ad}\overline{P}$. Now the curvature of $\mathbf{A}$ is

$$F_{\mathbf{A}} = F_A \; + \; d_A\sigma_j \wedge dx^j \; + \; \frac{1}{2}[\sigma_j, \sigma_k]\, dx^j \wedge dx^k,$$

which is a sum of orthogonal terms, and so

$$-\frac{1}{2}|F_{\mathbf{A}}|^2 = -\frac{1}{2}|F_A|^2 \; + \; \sum_j \frac{1}{2}|d_A\sigma_j|^2 \; - \; \frac{1}{2}\sum_{j<k} |\,[\sigma_j, \sigma_k]\,|^2.$$

Thus the dimensional reduction of the Yang-Mills lagrangian (4.5) has the canonical kinetic terms for the connection A and the scalar fields σ_j and a potential energy term

$$V = \frac{1}{2}\sum_{j<k} |\,[\sigma_j, \sigma_k]\,|^2. \tag{5.5}$$

Sometimes the real scalar fields σ_j are written in terms of complex scalar fields. For example, if $r = 2$ we set

$$\sigma = \frac{\sigma_1 + \sqrt{-1}\sigma_2}{\sqrt{2}};$$

it is a scalar field with values in the complexified adjoint bundle $\operatorname{ad}_{\mathbf{C}} P$. Then the potential energy (5.5) is

$$V = \frac{1}{2}\,\|\,[\sigma, \sigma]\,\|^2. \tag{5.6}$$

Our convention is that the double bar notation '$\|\cdot\|$' denotes the norm built from the inner product space on a hermitian vector space. We always use single bars for quadratic forms built from bilinear forms.

Quantum particles in $n = 4$ dimensions

As we indicated in Lecture 3, the quantization of free classical field theories leads to a one-particle Hilbert space which is a representation of an appropriate (super)symmetry group. Therefore, the representation theory of supersymmetry groups constrains the possible free classical field theories, and since nonfree theories can be regarded as perturbations of free theories, it constrains nonfree theories as well. Before launching into the construction of supersymmetric theories in $n = 4$ dimensions with $s = 4$ supersymmetries, we carry out the easy analysis of the representations of $P^{4|4}$.

As usual, we restrict our analysis to representations where only scalars (spin 0), spinors (spin 1/2), and vectors (spin 1) occur. First, without supersymmetry we seek representations of the Poincaré group P^4, as discussed in Lecture 1. In the massless case the little group is Spin(2). The irreducible complex representations are one-dimensional and indexed by the *helicity* $j \in \frac{1}{2}\mathbb{Z}$. (The helicity is the spin in the direction of the motion.) But recall that physical representations are constrained by CPT invariance, which in even dimensions means that the representation of the little group is real or quaternionic. Here we see that if helicity $j \neq 0$ occurs, then so does helicity $-j$ with the same multiplicity. In other words, the helicities are symmetric about 0. In the massive case the little group is Spin(3). Now all representations are real or quaternionic, so CPT gives no restriction. We decompose (complex) representations under a maximal torus Spin(2) $\subset$ Spin(3) into one-dimensional representations indexed by the *spin* $j \in \frac{1}{2}\mathbb{Z}$.

	-1	$-1/2$	0	$1/2$	1
massless scalar			1		
massive scalar			1		
massless spinor		1		1	
massive spinor		1		1	
massless vector	1				1
massive vector	1		1		1

Table 6: Particle representations of P^4

The various possibilities are summarized in Table 6. In the columns we indicate the dimension of the helicity or spin j representation of Spin(2). CPT requires that the table be symmetric about the vertical line $j = 0$. The most noteworthy feature is the difference between the massive and massless vector: a massive vector has 3 degrees of freedom whereas a massless vector has 2 degrees of freedom. The massless vector representation does not deform to the massive vector representation; rather, the sum of the massless vector and massless scalar deforms to a massive vector. In field theory this deformation is realized via the Higgs mechanism.

As discussed in Lecture 3, in the supersymmetric case we must represent a Clifford algebra as well. For $P^{4|4}$ this algebra has 4 generators in the massive case and 2 generators in the massless case. (The quadratic form is degenerate in the massless case leading to a Clifford algebra with 2 generators.) The minimal complex Clifford module $W = W^0 \oplus W^1$ has $\dim W^0 = \dim W^1 = 2$ in the massive case and $\dim W^0 = \dim W^1 = 1$ in the massless case. There must be an intertwining representation of the little group, and from the bracketing relations in the algebra $\mathfrak{p}^{4|4}$ it is easy to see that the spins which occur in the massive case are j, $j+1/2$, $j+1/2$, and $j+1$. In the massless case the helicities are j and $j+1/2$. Finally, we impose spin-statistics and require that W^0 consist of representations with integer spin or helicity and that W^1 consist of representations with half-integer spin or helicity. The possibilities are summarized in Table 7. There is no multiplet with a single real scalar; the minimal multiplet with scalar particles has 2 scalars. Its name *chiral*

multiplet derives from the description in terms of fields, which we discuss presently. We learn that this description must involve 2 real scalar fields, or equivalently one complex scalar field. The chiral multiplets and the massless vector multiplet each include a spinor with 2 physical degrees of freedom. So we expect the chiral and massless vector multiplets to include a single spinor field in the field description. We will encounter the massive vector multiplet through a supersymmetric version of the Higgs mechanism.

	-1	$-1/2$	0	$1/2$	1
massless chiral multiplet		1	2	1	
massive chiral multiplet		1	2	1	
massless vector multiplet	1	1		1	1
massive vector multiplet	1	2	2	2	1

Table 7: Multiplets for $P^{4\,4}$

The superspacetime $M^{4|4}$

As indicated in Table 1, the 4-dimensional real spin representation S of $\mathrm{Spin}(1,3)$ decomposes under complexification as

$$S_{\mathbb{C}} \cong S' \oplus S'', \qquad S'' \cong \overline{S'}.$$

Each of S', S'' has complex dimension 2 and each carries an invariant complex volume form ϵ. We build superspacetime using ΠS^*, and so fix complex odd linear coordinates θ^1, θ^2 on $\Pi S'^*$. We use the conjugate coordinates $\bar\theta^{\dot 1}, \bar\theta^{\dot 2}$ on $\Pi S''^*$. We use the notation '$\bar\theta^{\dot a}$' for the conjugate coordinates; the use of dotted indices is analogous to the use of barred indices on complex manifolds. Recall that the vector space V underlying Minkowski spacetime has complexification

$$V_{\mathbb{C}} \cong S'^* \otimes S''^*, \tag{5.7}$$

and so it is natural to use complex coordinates $y^{a\dot b}$ related to the odd complex coordinates $\theta^a, \theta^{\dot b}$ by (5.7). They are related to the standard x coordinates by the formulas

$$\begin{aligned} y^{1\dot 1} &= \frac{x^0 + x^1}{2}, \\ y^{2\dot 2} &= \frac{x^0 - x^1}{2}, \\ y^{1\dot 2} &= \frac{x^2 + \sqrt{-1}\,x^3}{2}, \\ y^{2\dot 1} &= \frac{x^2 - \sqrt{-1}\,x^3}{2}. \end{aligned}$$

There are nice formulas for the metric, wave operator, Dirac operator, star operator, etc. in this coordinate system.

As always we work in terms of a global framing rather than global coordinates, but here we use *complex* vector fields. The even elements of the framing are the bi-invariant vector fields

$$\partial_{a\dot{b}} = \frac{\partial}{\partial y^{a\dot{b}}}$$

and the odd elements are the left-invariant vector fields

$$\begin{aligned} D_a &= \frac{\partial}{\partial \theta^a} - \bar{\theta}^{\dot{b}} \partial_{a\dot{b}}, \\ \overline{D}_{\dot{a}} &= \frac{\partial}{\partial \theta^{\dot{a}}} - \theta^b \partial_{b\dot{a}}. \end{aligned} \tag{5.8}$$

The nontrivial bracket relation is

$$[D_a, \overline{D}_{\dot{b}}] = -2\partial_{a\dot{b}}.$$

The super Poincaré symmetries are generated by right invariant vector fields as before. We have the inclusion

$$i\colon \check{M}^4 \hookrightarrow M^{4|4}.$$

We assume that our coordinates are chosen so that the volume forms ϵ satisfy (4.13).

Note that our conjugation convention (1.11) makes the two equations in (5.8) really look conjugate! The conventions in many physics references lead to different looking formulas.

Supersymmetric σ-model on $M^{4|4}$

Our discussion of the representation theory of $P^{4|4}$ makes clear that we cannot write a sensible (free) theory using a real superfield $M^{4|4} \to \mathbb{R}$, since then there would be only one real scalar component field. Rather, we expect to use a complex superfield

$$\Phi\colon M^{4|4} \longrightarrow \mathbb{C}. \tag{5.9}$$

An unconstrained complex superfield would lead to too many component fields. For example, the restriction of first derivatives to $\check{M}^4$ would give both a complex spinor field in S' and an independent complex spinor field in S''. Upon quantization of a canonical free lagrangian we would obtain two spinor particles, whereas Table 7 indicates that we should only find one spinor particle in a multiplet with scalars. Therefore, we expect to put a constraint on the superfield (5.9). It turns out that the appropriate constraint is that we require Φ to be *chiral*, which means that its derivatives along the left-invariant distribution spanned by the $\overline{D}_{\dot{a}}$ vanish:

$$\overline{D}_{\dot{a}}\Phi = 0. \tag{5.10}$$

In this case $\overline{\Phi}$ is called *antichiral*. Chiral superfields on $M^{4|4}$ are analogous to holomorphic functions on a complex manifold; (5.10) is analogous to the ∂ equation. Sums and products of chiral superfields are chiral. Let $\mathcal{C}^*$ denote the sheaf of nonzero chiral superfields. Then

$$H^1(M^{4|4}, \mathcal{C}^*) = 0. \tag{5.11}$$

This is easy to see in components, for example.

The component fields of a chiral superfield are defined analogously to (4.14):

$$\begin{aligned}\phi &= i^*\Phi \\ \psi_a &= \frac{1}{\sqrt{2}} i^* D_a \Phi \\ F &= -\frac{1}{2}\epsilon^{ab} i^* D_a D_b \Phi\end{aligned}$$

Whereas in $M^{3\,2}$ the scalar multiplet consists of real fields, here the *chiral multiplet* $\{\phi, \psi, F\}$ consists of complex fields. In particular, there is a complex auxiliary field F. The free lagrangian in superspacetime is

$$\mathcal{L} = |d^4x|\, d^4\theta \left\{\frac{1}{4}\Phi\overline{\Phi}\right\}. \tag{5.12}$$

When we expand in components we obtain

$$L = \left\{\|d\phi\|^2 + \psi \not{D} \psi + \|F\|^2\right\} |d^4x|. \tag{5.13}$$

- As noted, F is an auxiliary field and in this simple model its (algebraic) equation of motion is $F = 0$. In the future we eliminate auxiliary fields from component lagrangians. Also, we will usually only write the bosonic part of the component lagrangian.
- Our definition of the component lagrangian (4.20) in this case differs from integration over $d^4\theta$, as anticipated in (4.19). We use the string of derivatives on the left hand side of the equation

$$\frac{1}{2} i^* (D_1 \overline{D}_{\dot{1}} \overline{D}_{\dot{2}} D_2 + D_2 \overline{D}_{\dot{2}} \overline{D}_{\dot{1}} D_1) = \int d^4\theta \; + \; \Box i^*$$

  to define L, and it differs from integration by the exact term indicated on the right hand side.

The *nonlinear* σ-model is more interesting. We first replace X by a complex manifold. It is easy to define a chiral map

$$\Phi\colon M^{4|4} \longrightarrow X.$$

Namely, Φ is chiral if $D_a\Phi$ is a vector field of type $(1,0)$ on X. Equivalently, Φ is chiral if the pullback of every local holomorphic function on X is chiral in the sense previously defined. Now to write the σ-model lagrangian we need to impose a Riemannian structure on X. In fact, the model is defined for X a *Kähler* manifold. To see why X has to be Kähler, we might try to relax some part of the Kähler structure (e.g. take a hermitian manifold whose $(1,1)$-form is not closed) and then rediscover the Kähler structure by imposing supersymmetry in the component lagrangian. Or, we could dimensionally reduce to $n = 3$ dimensions and ask that the σ-model, which always has $s = 2$ supersymmetries, carry $s = 4$ supersymmetries. We content ourselves here with simply defining the model *assuming* X is Kähler.

In fact, we make the simplifying assumption that X is *Hodge*. This means that X carries a holomorphic hermitian line bundle $L \to X$ with curvature

$$\operatorname{curv}(L) = \sqrt{-1}\,\omega,$$

where ω is the Kähler form. Now if $\Phi\colon M^{4|4} \to X$ is chiral, then there exist nonzero chiral sections $\tilde{\Phi}$ of $\Phi^* L \to M^{4|4}$. This follows from (5.11). The superspacetime lagrangian is written in terms of such a lift of Φ:

$$\mathcal{L} = |d^4x|\, d^4\theta\, \frac{1}{2} \log \|\tilde{\Phi}\|^2. \tag{5.14}$$

- The *action*—the integral of the lagrangian density—only depends on Φ, not on the lift $\tilde{\Phi}$. This is familiar in other field theories, e.g. a Wess-Zumino-Witten term in a σ-model. Our situation is better since the component lagrangian, which is a partial integral of the lagrangian density, is also independent of the lift.
- Usually one writes instead of (5.14) a local formula in terms of a Kähler potential K:

$$\mathcal{L} = |d^4x|\, d^4\theta\, \frac{1}{2} K(\overline{\Phi}, \Phi). \tag{5.15}$$

  Then a simple argument shows that the component lagrangian is independent of the choice of K. This expression has the disadvantage that it is only local, but the advantage that it is defined for Kähler manifolds which are not Hodge.
- The component lagrangian is similar to (5.13), but now there is a curvature term as in (4.23). Since we will not need the explicit formula, we omit it. The bosonic part is the standard σ-model with target X.

The superpotential

Suppose

$$W\colon X \longrightarrow \mathbb{C}$$

is a global holomorphic function. Then $\Phi^*(W)$ is a chiral superfield on $M^{4|4}$. It turns out that

$$|d^4x|\, d^2\theta\, \Phi^*(W) \tag{5.16}$$

is a super Poincaré-invariant density, but on a superspace $M^{4|2\mathbb{c}}$ whose odd directions are *complex* and modeled on $\Pi S'^*$. Furthermore, we can integrate out $d^2\theta$ using the expression

$$i^*(-\frac{1}{2}\epsilon^{ab} D_a D_b) = \int d^2\theta,$$

and the result is supersymmetric.

The total superspacetime lagrangian of the σ-model with superpotential W is

$$\mathcal{L} = |d^4x| \left\{ d^4\theta\, \frac{1}{2} \log \|\overline{\Phi}\|^2 \;+\; d^2\theta\, \frac{1}{2}\Phi^*(W) \;+\; d^2\theta\, \frac{1}{2}\overline{\Phi^*(W)} \right\}.$$

- We include the complex conjugate of (5.16) so that the lagrangian is real, as it must be in Minkowski spacetime.
- There are 3 different superspacetimes at work in this super lagrangian.
- The auxiliary field has a nontrivial field equation

$$F = -\phi^* \operatorname{grad} W.$$

- As in the σ-model on $M^{3|2}$ with potential, in this model with superpotential there is a scalar potential energy term

$$V = \| \operatorname{grad} W \|^2 \tag{5.17}$$

  and also a fermion mass term built from the Hessian of W.
- The analysis of the theory—its vacua, dimensional reductions, etc.—is entirely parallel to the 3-dimensional real story in Lecture 4.
- The R-symmetry of $M^{4|4}$ is broken by a generic superpotential W, but in certain cases one can assign a transformation law to Φ so that the theory with superpotential has R-symmetry.

Gauge theory on $M^{4|4}$

Fix a compact Lie group G and a bi-invariant inner product $\langle \cdot, \cdot \rangle$ on its Lie algebra $\mathfrak{g}$. As asserted in Lecture 4, a gauge field in a minimal supersymmetric gauge theory is a connection $\mathcal{A}$ constrained to be flat along the left-invariant odd distribution (see (4.38)). We remark that on $M^{4|4}$ the description of the unconstrained field is a bit more involved than on $M^{3|2}$, and we omit it. Again the basic invariant of a constrained connection is the even-odd curvature

$$W_a = \frac{1}{4} \epsilon^{\dot{b}\dot{c}} [\nabla_{a\dot{b}}, \overline{\mathcal{D}}_{\dot{c}}],$$

which is a section of the complexified adjoint bundle. The Bianchi identity implies that W_a is chiral. There are further ramifications of the Bianchi identities for the superspacetime curvature; in particular, they express the entire curvature in terms of W_a. The component fields are

$$\begin{aligned} A &= i^* \mathcal{A}, && \text{connection on a bundle } P \to \check{M}^4; \\ \lambda_a &= i^* W_a, && \lambda \in C^\infty\big(\check{M}^4, \Pi(\operatorname{ad}_{\mathbb{C}} P \otimes S')\big); \\ \sqrt{-1}\, E &= \frac{1}{2} \epsilon^{ab} i^* \mathcal{D}_a W_b, && E \in C^\infty(\check{M}^4, \operatorname{ad} P). \end{aligned}$$

- The set $\{A, \lambda, E\}$ of fields is called the *vector multiplet*. Here E is a *real* auxiliary field. (*Real* means that E takes values in the Lie algebra of the compact gauge group G, or rather in the real adjoint bundle.)
- Usually 'D' is used in place of 'E', but we prefer not to have yet another dee.
- The symmetric part of $i^* \mathcal{D}_a W_b$ is the self-dual part of the curvature of A.
- The supersymmetry transformation laws for the component fields are easily derived by applying (4.16).

The fundamental lagrangian has a *complex* coupling constant τ, which is usually written in terms of real coupling constants g^2, θ:

$$\tau = \frac{\theta}{2\pi} + \frac{4\pi\sqrt{-1}}{g^2}.$$

We require that $g^2 > 0$, or equivalently that τ live in the upper half plane. The lagrangian in superspacetime is

$$\mathcal{L} = |d^4x| \left\{ -\frac{1}{16\pi} \operatorname{Im}\Big(d^2\theta\, \tau\, \epsilon^{ab} \langle W_a, W_b \rangle \Big) \right\}. \tag{5.18}$$

The expansion into a component lagrangian is

$$L = \frac{1}{g^2}\left\{ -\frac{1}{2}|F_A|^2 + \langle \bar\lambda \not{D}_A \lambda \rangle + \frac{1}{2}|E|^2 \right\} |d^4x| + \frac{\theta}{16\pi^2} \langle F_A \wedge F_A \rangle.$$

- Notice that the superspacetime lagrangian (5.18) involves only $d^2\theta$ terms, no $d^4\theta$ terms. By contrast, there are $d^4\theta$ terms in the kinetic term (5.14) for the chiral superfield.
- We eliminate the auxiliary field E by its equation of motion $E = 0$.
- For $\theta = 0$ the component lagrangian is a density and does not require a choice of orientation. Thus it is invariant under orientation-reversing Poincaré transformations. But for $\theta \neq 0$ we need an orientation, and so the θ-term (which we anticipated in (5.4)) breaks CP invariance.
- In the quantum theory the integrand of the functional integral is $e^{iL/\hbar}$, and then the integrality of characteristic classes implies that θ is a periodic variable.
- We wrote the theory assuming the gauge group G is simple. Otherwise, we have coupling constants for each simple or abelian factor. We previously absorbed the coupling constant into the inner product $\langle \cdot, \cdot \rangle$ on $\mathfrak{g}$, but here it is more natural to normalize the inner product so that θ is periodic with period 2π.

The general theory on $M^{4|4}$

The most general supersymmetric theory (without gravity) includes both a gauge superfield and a chiral superfield. We assume the chiral superfield takes values in a Hodge manifold to simplify the presentation; as before there is also a formulation for Kähler targets.

The data which defines the theory are

(5.19)	G	compact Lie group
	$\langle \cdot, \cdot \rangle$	bi-invariant scalar product on $\mathfrak{g}$
	$L \to X$	Hodge manifold on which G acts
	$W\colon X \longrightarrow \mathbb{C}$	holomorphic function invariant under G

The action of G on X is by automorphisms of the Kähler structure, and we assume we are given a lift to an action on L. The reader should compare with the data (4.7) required in the most general bosonic theory. The fields in the model are

$\mathcal{A}$	connection on a principal G bundle $\mathcal{P} \longrightarrow M^{4\,4}$
$\Phi\colon \mathcal{P} \longrightarrow X$	chiral equivariant map

Of course, Φ can be viewed as a chiral section of a bundle over $M^{4|4}$ with typical fiber X. To write the lagrangian we lift to a nonzero equivariant chiral map $\tilde{\Phi}\colon \mathcal{P} \to L$ as before. Now the general lagrangian we want to consider is

$$\mathcal{L} = |d^4x|\Big\{d^4\theta\ \frac{1}{2}\log\|\tilde{\Phi}\|^2 + \mathrm{Re}\Big[d^2\theta\ \Big(\frac{\sqrt{-1}\,\tau}{16\pi}\epsilon^{ab}\langle W_a, W_b\rangle + \Phi^* W\Big)\Big]\Big\}. \tag{5.20}$$

A new feature enters into the component lagrangian: the *moment map*

$$\mu\colon X \longrightarrow \mathfrak{g}^*$$

of the action of G on X. It is defined by the lift of the action to L:

$$\mu(\zeta) = -\sqrt{-1}\ \mathrm{vert}_1(\hat{\zeta}), \qquad \zeta \in \mathfrak{g},$$

where $\hat{\zeta}$ is the vector field on L induced by ζ, and we take the vertical part of $\hat{\zeta}$ at points of unit norm. Then after eliminating auxiliary fields the bosonic part of the component lagrangian is

$$\begin{aligned} L_{\mathrm{bos}} = \Big\{\Big(-\frac{1}{2g^2}|F_A|^2 + \|d_A\phi\|^2\Big) - \phi^*\big(\|\operatorname{grad} W\|^2 + 2g^2|\mu|^2\big)\Big\}\,|d^4x| \\ + \frac{\theta}{16\pi^2}\langle F_A \wedge F_A\rangle. \end{aligned} \tag{5.21}$$

- The potential function

$$V = \|\operatorname{grad} W\|^2 + 2g^2|\mu|^2 \tag{5.22}$$

  has contributions from both the superpotential and the moment map. The first term comes from the complex auxiliary field F, the second from the real auxiliary field E. As expected in a supersymmetric theory, V is nonnegative.
- For a *fundamental* lagrangian we require renormalizability. This means that X is a complex inner product space and W a polynomial of degree at most 3.
- For an *effective* lagrangian we replace the constant τ in (5.20) by a holomorphic function

$$\tau\colon X \longrightarrow \mathbb{C}. \tag{5.23}$$

  Recall that τ must take values in the upper half plane. In the bosonic lagrangian we then find that both g^2 and θ are functions of ϕ.
- The moduli space of vacua is

$$\mathcal{M}_{\mathrm{vac}} = \mu^{-1}(0) \cap \mathrm{Crit}(W)\ /\ G.$$

  When $W = 0$ it is the Kähler (symplectic) quotient $X//G$. In general, W descends to a holomorphic function $\overline{W}$ on the Kähler quotient and the moduli space of vacua is the set of critical points of $\overline{W}$ on this quotient. We can often identify it with the set of critical points of W on the quotient of X by the complexification of G.
- We can always shift the moment map by a character $G \to \mathbb{T}$. This holds even if there is no chiral superfield—X is a point. In that case the basic data (5.19) include a lift of the G action to the Hodge line $\mathbb{C} \to \mathrm{pt}$, which is precisely a character of G. The corresponding term in the action, a constant which is twice the norm square of the infinitesimal character, is called a *Fayet-Iliopoulos* term.

Example (super Higgs mechanism). As a simple example to illustrate the Fayet-Iliopoulos term, take $G = \mathbb{T}$ to be the circle group and $X = \mathbb{C}$ with its standard Kähler structure and the circle group acting by multiplication. Then one possible moment map is $\mu(z) = \frac{1}{2}|z|^2$. We take $W = 0$. So the potential energy is $V = \frac{1}{2}|z|^4$, which has a *degenerate* zero at the origin. The moduli space of vacua consists of a single point at which there is a massless gauge field and a massless complex scalar. Now we shift the moment map by a character:

$$\mu(z) = \frac{1}{2}|z|^2 - \frac{a^2}{2}, \qquad a \in \mathbb{R}.$$

Then the new potential is the quartic

$$\frac{1}{2}(|z|^2 - a^2)^2.$$

The moduli space of vacua is the set of zeros of V divided by the $\mathbb{T}$ action, so again a single point. But now the Higgs mechanism is in play and there are no massless scalar fields or massless gauge fields.

We have not written the fermionic part of the lagrangian, but a similar Higgs phenomenon occurs with the spinor fields in this model.

Dimensional reductions of theories with $s = 3$ and $s = 4$ supersymmetries

In Lecture 4 we considered in detail the dimensional reduction of the supersymmetric σ-model with symmetry group $P^{3|2}$. The scalar superfield $\Phi\colon M^{3|2} \to X$ reduces to a scalar superfield on the superspacetime $M^{n|2}$ with $n = 2$ or $n = 1$. The supersymmetric σ-model on $M^{4|4}$ reduces similarly. There are superspacetimes $M^{n|4}$ with $n < 4$ and a chiral superfield $\Phi\colon M^{4|4} \to X$ reduces to a chiral superfield on the reduced superspacetime. For curved target manifolds the quantum theory is only well-defined if $n = 2$ or $n = 1$. The theory on $M^{2\,(2,2)}$ has been much studied; in the classical version the BPS story leads to the equations of holomorphic Morse theory. The Hilbert space of supersymmetric quantum mechanics on $M^{1|4}$ is the space of complex differential forms on the Kähler manifold X, and the 4 supercharges are written in terms of the $\bar{\partial}$ operator and its conjugate and adjoint. The supersymmetry group $P^{1|4}$ and its R-symmetry group $SU(2)$ are represented on the complexified forms. In particular, the action of the R-symmetry group incorporates the Kähler identities. Notice that the group of R-symmetries is the double cover of the group of spatial rotations in $P^{4|4}$ acting by outer automorphisms on the subgroup $P^{1|4}$. Thus we have a geometric explanation of the Kähler identites using supersymmetry.

To this point we have not considered the dimensional reduction of a supersymmetric *gauge* theory. We explained earlier that in the bosonic case the reduced theory has scalar fields coming from spatial components of the connection in higher dimensions. So in a supersymmetric gauge theory we learn that a vector multiplet with extended supersymmetry—that is, the dimensional reduction of the maximal dimensional vector multiplets we have been discussing—contains scalar fields. Since supersymmetry links these scalar fields to a connection, we expect that at least in some cases the geometry of these scalars in an effective theory is constrained in a

different manner than that of scalar fields which are in scalar multiplets and not linked to connections. The geometry of the scalars in a vector multiplet with extended supersymmetry goes by the generic name *special geometry*. We mention in passing that scalar fields in theories with *local* supersymmetry (supergravity) are also constrained, and the constraints lead to different geometric structures. In both local and global supersymmetry, the geometry depends strongly on the dimension of spacetime and the amount and type of supersymmetry.

Supersymmetric lagrangians with both vector superfields and scalar superfields include a wide variety of terms. The few examples we discuss serve only as an introduction, not as an exhaustive catalog.

As a first example, consider pure supersymmetric gauge theory on $M^{3|2}$ reduced to $M^{2|(1,1)}$. From the component description (4.40) we see that the reduced vector multiplet has a real scalar field, a spinor field of each chirality, and a connection. Since there is a superspacetime description in $n = 3$ dimensions, we obtain such a description in $n = 2$ dimensions. The field is a connection $\mathcal{A}$ on $M^{2|(1,1)}$, but the constraint is no longer simply (4.38)—we allow one component of the curvature along the odd distribution to be nonzero. Namely, in the superspacetime $M^{2\,(1,1)}$ we use a framing by real left invariant vector fields—two odd vector fields D_+, D_- and two even vector fields ∂_+, ∂_-—with brackets (see (3.14)):

$$\begin{aligned} [D_+, D_+] &= -2\partial_+ \\ [D_-, D_-] &= -2\partial_- \\ [D_+, D_-] &= 0, \end{aligned} \tag{5.24}$$

We still demand zero curvature in the D_+, D_+ and $D\ \ , D_-$ directions, but allow a nonzero bracket for the covariant derivatives in the D_+, D_- directions:

$$[\mathcal{D}_+, \mathcal{D}_-] = 2\Sigma \tag{5.25}$$

for some adjoint-valued scalar superfield Σ. The restriction of Σ to $\check{M}^2$ is the adjoint-valued scalar field of the vector multiplet. The Bianchi identities allow us to write all components of the curvature in terms of Σ. We find that the reduction of the superfield lagrangian (4.42) to $M^{2|(1,1)}$ is

$$\mathcal{L} = \frac{1}{2}\langle \mathcal{D}_+\Sigma, \mathcal{D}_-\Sigma\rangle\, |d^2x|\, d^2\theta.$$

This is precisely the σ-model lagrangian for a scalar superfield Σ; it is the dimensional reduction of (4.18) to $M^{2|(1,1)}$. Where, then, is the connection A in the component formulation? The auxiliary component field F of the superfield Σ (see (4.14)) is (up to a sign) the Hodge star of the curvature of A. The bosonic lagrangian (4.23) reduces to the correct lagrangian in $\check{M}^2$ for the adjoint-valued scalar field.

Note the following:

- At least formally, the scalar superfield Σ is related to the central extension of the supersymmetry algebra: Compare (3.14) with (5.24) and (5.25).
- This example illustrates a general feature of vector superfields in superspacetimes with more than the minimal amount of supersymmetry: the curvature constraint allows nonzero components of the curvature along the odd distribution.

An effective lagrangian for a scalar superfield Σ on $M^{2|(1,1)}$ which is at most quadratic in first derivatives can be written for a general metric on the manifold X where Σ takes its values. But if we write such an effective lagrangian for the vector superfield on $M^{2|(1,1)}$ described here, then we cannot allow a general G-invariant metric on the Lie algebra $X = \mathfrak{g}$ where Σ takes its values. Otherwise, the term in the component lagrangian with the auxiliary field, which here is the kinetic term for the connection A, might not be quadratic in the curvature F_A. Rather, the most general metric we can allow on $\mathfrak{g}$ is a metric conformally equivalent to the constant metric. We can include a potential term as well. So the most general effective superspacetime lagrangian to the order in the derivative expansion we are considering is

$$\mathcal{L}_{\text{eff}} = \left\{ \frac{1}{2g^2(\Sigma)} \langle \mathcal{D}_+\Sigma, \mathcal{D}_-\Sigma \rangle + \Sigma^*(h) \right\} |d^2x|\, d^2\theta,$$

where $g^2 \colon \mathfrak{g} \to \mathbb{R}^{>0}$ and $h \colon \mathfrak{g} \to \mathbb{R}$ are arbitrary functions. Notice that the quadratic function $h = \langle \Sigma, \Sigma \rangle$ gives in the component lagrangian a term proportional to (5.3). So (5.3) does appear in supersymmetric lagrangians.

In $n = 2$ dimensions a gauge field has no physical degrees of freedom, and so we may as well write an effective lagrangian simply in terms of the scalar superfield Σ, forgetting the vector superfield $\mathcal{A}$. Then we can have as target an arbitrary Riemannian manifold X and an arbitrary potential function. This simple example shows that the identification of the correct physical degrees of freedom dramatically affects the set of possible effective lagrangians. In one case the manifold of scalars is the Lie algebra $\mathfrak{g}$ with a conformally trivial metric; in the other it is an arbitrary Riemannian manifold.

It is instructive to consider the effective lagrangian (to the order we are considering) for a model with a vector superfield $\mathcal{A}$ and a scalar superfield Φ with values in a manifold Y. Then the manifold of scalars is $X = Y \times \mathfrak{g}$, and the Riemannian metric is constrained to be of the form

$$ds^2_X(\phi, \sigma) = ds^2_Y(\phi) + \frac{1}{g^2(\phi, \sigma)} \langle \cdot, \cdot \rangle.$$

Next, we ask a general question: Are there nontrivial connections on a superspacetime $M^{n|s}$ invariant under the supersymmetry group $P^{n|s}$? The answer to the analogous question on $\check{M}^n$ is no—by Lorentz invariance all components of the curvature vanish. This Lorentz invariance argument also shows that for a $P^{n|s}$-invariant connection on $M^{n|s}$ the even-even and even-odd components of the curvature vanish, but it says nothing about the odd-odd components. The translational invariance says that the odd-odd components are covariant constant. Hence a $P^{n|s}$-invariant connection has odd-odd curvature $[\mathcal{D}_a, \mathcal{D}_b] = 2\Sigma_{ab}$ for $\{\Sigma_{ab}\}$ a collection of *commuting* elements of the gauge algebra. (The covariant constancy implies that the Σ_{ab} pairwise commute.)

In supersymmetric theories we use *constrained* connections. With minimal supersymmetry ($M^{2|(1,0)}$, $M^{3|2}$, $M^{4|4}$, $M^{6|(8,0)}$) the constraint says that all odd-odd components of the curvature vanish, and so every super Poincaré invariant constrained connection is trivial. But as we have seen, with extended supersymmetry some odd-odd components can be nonzero and in that case there are nontrivial super Poincaré invariant connections. If we treat such a connection as fixed (a

background connection) and couple a variable scalar superfield to it, then we can write supersymmetric terms for the scalar superfield which end up involving additional masses related to the Σ_{ab}. Such terms are not dimensional reductions.

As a concrete example, consider a theory on the superspacetime $M^{3|4}$, which we view as a quotient of $M^{4|4}$. We use the odd left-invariant vector fields (5.8), only now we have $\partial_{1\dot{2}} = \partial_{2\dot{1}}$. A constrained connection has all odd-odd components of curvature vanishing except

$$\operatorname{Im}[\mathcal{D}_1, \mathcal{D}_{\dot{2}}] = 2\Sigma$$

for some *real* adjoint-valued scalar superfield Σ. The Bianchi identity gives constraints on Σ. A connection with Σ constant and all other components of curvature vanishing is super Poincaré invariant. Now let a chiral superfield Φ take values in an associated hermitian vector bundle. Then the supersymmetric kinetic term (see (5.12))

$$\mathcal{L} = \frac{1}{4}\|\Phi\|^2 \, |d^3x| \, d^4\theta$$

gives in components

$$L = \|d\phi\|^2 + \langle \bar{\psi} \not{D} \psi \rangle + \|\sigma \cdot \phi\|^2 + \langle \psi, \sigma \cdot \psi \rangle,$$

where σ is the constant value of Σ. So we have *real* mass terms. (Note that complex mass terms for Φ are easily obtained from a superpotential (5.16).) Such terms are not dimensional reductions from 4 dimensions, nor can they be written in superspacetime just in terms of Σ, Φ (without the background connection).

This construction works in other dimensions and always gives mass terms. However, the $M^{3\,4}$ example in the previous paragraph is the only one I know where there is no alternative simpler superspacetime expression.

Finally, consider the reduction of the supersymmetric gauge theory on $M^{4\,4}$ to $M^{2|(2,2)}$. Now we obtain two real adjoint-valued scalars, or equivalently one complex adjoint-valued scalar. These have a superfield description analogous to (5.25). Namely, in the superspace $M^{2\,(2,2)}$ we use a framing by complex vector fields $D_+, \overline{D}_+, D\ \ , \overline{D}\ \ , \partial_+, \partial_-$, and then define the $\mathfrak{g}_{\mathbb{C}}$-valued superfield Σ by

$$[\mathcal{D}\ \ , \overline{\mathcal{D}}_+] = -2\sqrt{2}\,\Sigma.$$

The Bianchi identities imply that Σ is a *twisted chiral superfield*: $\overline{\mathcal{D}}_+\Sigma = \mathcal{D}_-\Sigma = 0$. It turns out that the reduced lagrangian is an integral over $d^4\theta$ of a constant times the norm square of Σ. Now the complex auxiliary field of Σ includes both the Hodge star of the curvature of the reduced connection A and a real auxiliary field. When we solve for the auxiliary field we obtain the potential energy

$$V = \frac{1}{2}\| \, [\sigma, \bar{\sigma}] \, \|^2$$

as expected from (5.6). As before, we have two different stories for a possible effective lagrangian depending on whether we write it in terms of an arbitrary twisted chiral superfield or a vector superfield. (In the former case, we may as well take the field to be chiral and then the target is an arbitrary Kähler manifold with superpotential.)

There are more interesting possibilities in effective theories which also include a chiral superfield.

In these examples we do not meet any nontrivial special geometry for the scalars in the vector multiplet. For that we need models with more supersymmetry.

Theories with $s = 8$ supersymmetries

Here we simply survey the results and do not give many details. Recall from Table 5 that for the σ-model there is no known superspacetime model in general, so we cannot present the theory in a manifestly supersymmetric way. This means that the model in components has no auxiliary fields and that the supersymmetry algebra does not close off-shell. For the pure gauge theory there *is* a superspacetime model, but we do not formulate that here. The model in components has 3 real adjoint-valued auxiliary fields.

We begin our discussion in the maximal dimension $n = 6$, where $s = (8,0)$ supersymmetry is the minimal amount. Consider briefly the representations of $P^{6|(8,0)}$. The massless little group is Spin(4) and the Clifford algebra we must represent has 4 generators, so the minimal Clifford module is $W = W^0 \oplus W^1$ with $\dim W^0 = \dim W^1 = 2$. Each irreducible representation of Spin(4) is real or quaternionic, so CPT imposes no constraint. The minimal solution has W^0 consisting of 2 scalars and W^1 consisting of a chiral spinor. This multiplet is called a *half-hypermultiplet*. It is the first example we meet of a representation of a super Poincaré group which has no realization in terms of fields. The *hypermultiplet* is, not surprisingly, two half-hypermultiplets; it contains 4 scalars. It has a realization as a multiplet of fields which contains a quaternionic-valued scalar field. The little group representation for the *massless vector multiplet* has even part the 4-dimensional spin 1 representation of Spin(4) and odd part two copies of a chiral spin representation. A massive representation has little group Spin(5) $\cong Sp(2)$ and the restriction of any spin representation to Spin(4) $\cong Sp(1) \times Sp(1)$ contains spinors of both chiralities. It follows that the hypermultiplet, whose little group representation consists of chiral spinors, does not deform to a massive representation.

In the field representation, the *linear* hypermultiplet has a quaternionic-valued scalar field, as mentioned above. Just as there is a constraint on the target of a *nonlinear* σ-model with 4 supersymmetries—it must be Kähler –here we find that the target of nonlinear a σ-model with 8 supersymmetries must be *hyperkähler*. In $n = 6$ dimensions there is no supersymmetric potential term possible. This is consistent with the representation theory discussed above where we found that no mass term is possible. There is a supersymmetric gauge theory, as we have mentioned several times. The vector multiplet consists of a connection, two chiral spinor fields with values in the adjoint bundle, and three real auxiliary fields with values in the adjoint bundle. To write the coupled model we need:

G	compact Lie group
$\langle\cdot,\cdot\rangle$	bi-invariant scalar product on G
X	hyperkähler manifold on which G acts
$\mu_{\mathbb{H}} \colon X \longrightarrow \mathfrak{g}^* \otimes \operatorname{Im}\mathbb{H}$	hyperkähler moment map

The appearance of the moment map is not surprising given (5.21). The bosonic lagrangian (with coupling constant $g^2 = 1$) is then

$$L_{\text{bos}} = -\frac{1}{2}|F_A|^2 + \frac{1}{2}|d_A\phi|^2 - \phi^*\big(2|\mu_{\mathbb{H}}|^2\big),$$

with nonnegative potential energy

$$V = 2|\mu_{\mathbb{H}}|^2. \tag{5.26}$$

The moduli space of classical vacua is the *hyperkähler quotient*

$$\mathcal{M}_{\text{vac}} = \mu_{\mathbb{H}}^{-1}(0) \,/\, G.$$

The ambiguity in choosing the moment map $\mu_{\mathbb{H}}$ leads to Fayet-Iliopoulos terms as in theories with 4 supersymmetries.

We could consider the dimensional reduction in all dimensions ≤ 5, but for lack of time, space, and energy we consider only the reduction to $n = 4$ dimensions. These are theories invariant under $P^{4|8}$, the so-called $N = 2$ supersymmetry group in $n = 4$ dimensions. In this case a supersymmetric potential function does exist, and it is specified by a vector field ζ on the hyperkähler manifold X which is an infinitesimal automorphism of the hyperkähler structure. In the coupled model with gauge group G, we demand that G fix ζ. Then the supersymmetric lagrangian of the general form (5.6) (with fermions added) is determined by the data:

(5.27)

G	compact Lie group
$\langle\cdot,\cdot\rangle$	bi-invariant scalar product on $\mathfrak{g}$
X	hyperkähler manifold on which G acts
$\mu_{\mathbb{H}}\colon X \longrightarrow \mathfrak{g}^* \otimes \operatorname{Im}\mathbb{H}$	hyperkähler moment map
ζ	infinitesimal hyperkähler automorphism which commutes with G.

Now the scalar fields in the model are

ϕ	values in X
σ	values in $\mathfrak{g}_{\mathbb{C}}$.

Of course, both scalar fields take values in a twisted bundle determined by the connection A. Both scalar fields are complex. The bosonic lagrangian has the canonical kinetic terms and the potential energy is

$$V = \frac{1}{2}\|\,[\sigma,\bar\sigma]\,\|^2 \;+\; \|\,\sqrt{2}\,\hat\sigma_\phi + \zeta_\phi\,\|^2 \;+\; 2|\mu_{\mathbb{H}}|^2. \tag{5.28}$$

Here $\hat\sigma$ is the complex vector field on X induced by the action of $\sigma \in \mathfrak{g}_{\mathbb{C}}$.

- We met the first term of V in (5.6) and the last term in (5.26). The second term is new. Without $\sqrt{2}\,\hat\sigma_\phi$ it is analogous to (5.17); the appearance of $\sqrt{2}\,\hat\sigma_\phi$ becomes clear in the formulation on $M^{4|4}$ which we consider below.

- In a fundamental lagrangian the manifold X is a quaternionic vector space and the vector field ζ is a quaternionic linear transformation.
- We discuss effective lagrangians below.
- In general, the moduli space $\mathcal{M}_{\text{vac}}$ is singular with different branches and different long-range physics (e.g. massless spectrum) on each of the three branches. To find $\mathcal{M}_{\text{vac}}$ we set each of the terms in (5.28) to zero and divide by the action of G.

It is convenient to formulate the $P^{4\,8}$-invariant theories in terms of the superspace $M^{4|4}$. This makes the $P^{4|4}$ supersymmetry manifest. We use the terminology '$N = 2$' to refer to multiplets for $P^{4|8}$ and '$N = 1$' to refer to multiplets for $P^{4|4}$.

The $M^{4|4}$ superfield representation of an $N = 2$ $\mathbb{H}$-valued hypermultiplet is a pair (Φ_1, Φ_2) of $N = 1$ $\mathbb{C}$-valued chiral multiplets. Here we need to write $\mathbb{H} \cong \mathbb{C} \oplus \mathbb{C}$. Similarly, to write a hypermultiplet with values in a hyperkähler manifold X in terms of $N = 1$ chiral multiplets, we need to fix a distinguished Kähler structure on X. An $N = 2$ vector multiplet with gauge group G is represented by a pair $(\mathcal{A}, \Sigma)$ consisting of an $N = 1$ vector multiplet $\mathcal{A}$ with gauge group G and an $N = 1$ chiral multiplet Σ with values in $\mathfrak{g}_{\mathbb{C}}$. To formulate the general theory we need to modify the data (5.27) to fit the form (5.19). Thus we fix a distinguished Kähler structure, and decompose the moment map as

$$\mu_{\mathbb{H}} = (\mu_{\mathbb{R}}, \mu_{\text{hol}}).$$

Here μ_{hol} can be thought of as the moment map for the complexified $G^{\mathbb{C}}$ action, where X has the holomorphic symplectic form η given from the hyperkähler structure and the distinguished Kähler structure. The manifold of scalars is

$$\tilde{X} = X \times \mathfrak{g}_{\mathbb{C}}$$

and we have the Kähler moment map

$$\mu(\phi, \sigma) = \mu_{\mathbb{R}}(\phi) + \frac{\sqrt{-1}}{2}[\sigma, \bar{\sigma}].$$

The superpotential W we use is built out of the holomorphic moment map and the vector field ζ on X:

$$W = \mu_\zeta + \sqrt{2}\,\tilde{\mu}_{\text{hol}}, \tag{5.29}$$

where μ_ζ is the holomorphic moment map for the vector field ζ (in other words, $\iota(\zeta)\eta = -\partial\mu_\zeta$) and

$$\mu_{\text{hol}} \colon X \longrightarrow \mathfrak{g}_{\mathbb{C}}^*$$

is transposed to

$$\tilde{\mu}_{\text{hol}} \colon X \times \mathfrak{g}_{\mathbb{C}} \longrightarrow \mathbb{C}.$$

The formula for the scalar potential function (5.28) then follows from (5.22).

We encounter an interesting example of special geometry when we consider the effective lagrangian for an $N = 2$ vector multiplet. Let's take the gauge group to be the abelian group $G = \mathbb{T}^d$. Then the complexified Lie algebra is $\mathfrak{g}_{\mathbb{C}} \cong \mathbb{C}^d$, and this is the manifold on which the scalars take their values. In fact, we will find

that the description we give is local and only holds on an open set $U \subset \mathbb{C}^d$. The quantum moduli space of a theory is patched together from such local descriptions (and typically also has singularities). We write the effective lagrangian in terms of $N = 1$ superfields on $M^{4|4}$, as in the previous paragraph. Choosing the standard basis of $\mathbb{C}^d$ we decompose the $N = 1$ vector multiplet $\mathcal{A}$ into components $\mathcal{A}^i$. Then the most general effective lagrangian to the order we consider which is $P^{4|4}$ invariant was already written in (5.20) and (5.23). It is:

$$\mathcal{L}_{\text{eff}} = |d^4x| \left\{ d^4\theta \, \frac{1}{2} K(\Sigma, \overline{\Sigma}) \; - \; \mathrm{Im}\big(d^2\theta \, \frac{\tau_{ij}(\Sigma)}{16\pi} \epsilon^{ab} \langle W_a^i, W_b^j \rangle\big) \right\}.$$

Here K is a Kähler potential on U and τ is a holomorphic function from U to the upper half plane.

- K is constrained by the fact that the Kähler form is positive definite, and τ is constrained by the fact that it takes values in the upper half plane.
- We used the local form (5.15) for the first term.
- The most general $P^{4|4}$-invariant lagrangian includes a superpotential, but from (5.29) we see that there is no superpotential in a $P^{4|8}$-invariant lagrangian.

When we impose $N = 2$ supersymmetry (or $P^{4\,8}$-invariance), we find that K and τ are coupled. Namely, locally there exists a *holomorphic prepotential*

$$\mathcal{F} : U \longrightarrow \mathbb{C}$$

such that

$$K = \frac{1}{2} \mathrm{Im}\big(\frac{\partial \mathcal{F}}{\partial z^i} \bar{z}^i\big)$$

and

$$\tau_{ij} = \frac{\partial^2 \mathcal{F}}{\partial z^i \partial z^j}$$

in standard coordinates on $U \subset \mathbb{C}^d$. These are the defining equations of *special Kähler geometry*. The most common examples of special Kähler manifolds occur as the base space of an algebraic integrable system. Seiberg and Witten computed the low energy effective lagrangian of the model with $G = SU(2)$ (with and without hypermultiplets in the fundamental representation and adjoint representation). Subsequent work treated more general gauge groups and matter representations, and the connection with integrable systems emerged.

Finally, we mention that if we reduce the σ-model to $n = 1$ dimensions, then in the quantum theory we obtain a representation of the supersymmetry algebra and R-symmetries on the space of complex differential forms on a hyperkähler manifold. In particular, the R-symmetry group is Spin(5)—the double cover of the group of rotations in the 5 spatial directions in $\check{M}^6$.

Theories with $s = 16$ supersymmetries

We begin in the maximal dimension $n = 10$ with $P^{10|(16,0)}$-invariant theories. As for the representation theory, the massless little group is Spin(8), and the relevant Clifford algebra has 8 generators. So a minimal complex Clifford module $W \cong W^0 \oplus W^1$ has $\dim W^0 = \dim W^1 = 8$. The only solution with spins ≤ 1 is to take W^0

to be the vector representation of Spin(8) and W^1 a half-spinor representation of Spin(8). This is the vector multiplet and the field content is that of minimal supersymmetric Yang-Mills theory. So we can hope that the lagrangian

$$L = \{-\frac{1}{2}|F_A|^2 + \frac{1}{2}\langle \lambda \not{D}_A \lambda\rangle\}\, |d^{10}x| \tag{5.30}$$

is supersymmetric in the *nonabelian* case. That is indeed the case, as we illustrate below. In this theory λ is a *dual* spinor field with values in the adjoint bundle.

- There is no σ-model, as the representation theory indicates. That is the argument in Nahm's theorem which lies behind the corresponding blank entry in Table 5.
- There are also theories of supergravity in $n = 10$ dimensions (and $n = 11$ dimensions). Their field content can easily be determined by these representation theory arguments.
- The reduction of the $n = 10$ theory to any dimension $n \leq 10$ is the gauge theory with maximal supersymmetry. The $P^{4|16}$-invariant theory in $n = 4$ dimensions, called $N = 4$ *Yang-Mills*, is predicted to be a conformally invariant quantum field theory which exhibits exact $SL(2,\mathbb{Z})$ electromagnetic duality. Notice that there are 6 $\mathfrak{g}$-valued scalar fields, and we can organize them into 3 $\mathfrak{g}_{\mathbb{C}}$-valued scalar fields. If we write the classical theory in $M^{4|4}$, then there is a quartet of superfields $\{\mathcal{A}, \Sigma_1, \Sigma_2, \Sigma_3\}$ consisting of a vector superfield and 3 chiral superfields. The fundamental theory has the canonical lagrangian (5.20) with superpotential
$$W = \sqrt{2}\,\langle \Sigma_1, [\Sigma_2, \Sigma_3]\rangle.$$

We now show that (5.30) is supersymmetric. It is useful to consider the problem in general, meaning in n dimensions with s supersymmetries. Consider first the abelian theory, which is free. Upon quantization, according to the discussion at the end of Lecture 3, the gauge field A has $n-2$ physical degrees of freedom and the spinor field has $s/2$ physical degrees of freedom. With one exception the number of bosonic and fermionic degrees of freedom in a supersymmetric theory must be equal, from which $n = 2 + s/2$. This is satisfied only for $n = 3, 4, 6, 10$ with minimal supersymmetry $s = 2, 4, 8, 16$. Our analysis below applies to these cases; in all other cases the representation theory shows that (5.30) cannot be supersymmetric (in either the abelian or nonabelian case). This argument is in accord with Table 5, but there is an additional theory listed there in $n = 2$ dimensions with $s = (1,0)$ supersymmetry. This falls under the exception referred to above. In fact, with *any* chiral supersymmetry $s = (s^+, 0)$ in $n = 2$ dimensions, the left-handed supersymmetries act trivially on right-handed fields (which are annihilated by ∂_+). So there is an $s = (s^+, 0)$ supersymmetric multiplet with s^+ right-handed spinor fields. The classical gauge field A does not give rise to a quantum particle in 2 dimensions. Therefore, it is possible that (5.30) is supersymmetric in $n = 2$ dimensions with any (right or left) chiral supersymmetry.

Now we write (5.30) in more detail. We work in n-dimensional Minkowski spacetime and fix a real spin representation S. Thus we have pairings (3.1), (3.2) which satisfy the Clifford relation (3.3). The spinor field λ takes values in the *dual* space S^*. The lagrangian is

$$L = \big(-\frac{1}{4}g^{\mu\sigma}g^{\nu\rho}\langle F_{\mu\nu}, F_{\sigma\rho}\rangle + \frac{1}{2}\Gamma^{\sigma}_{ab}\langle \lambda^a, D_\sigma \lambda^b\rangle\big)\, |d^n x|, \tag{5.31}$$

where $\langle\cdot,\cdot\rangle$ is the invariant inner product on the gauge algebra, the covariant derivative is $D_\sigma = \partial_\sigma + A_\sigma$, and the curvature is $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu]$. We want a theory with supersymmetry algebra (3.11), where the bracket of the odd elements is (3.9). Extrapolating from the three-dimensional case (4.41), we postulate the action of the even vector field $\eta^a Q_a$ on the fields to be

$$\begin{aligned}(\hat{\zeta}A)_\mu &= \eta^a g_{\mu\nu}\Gamma^\nu_{ab}\lambda^b \\ \hat{\zeta}\lambda^a &= \frac{1}{2}\eta^b\tilde{\Gamma}^{\mu ac}\Gamma^\nu_{cb}F_{\mu\nu}.\end{aligned} \tag{5.32}$$

Each side of the first equation is a (component of a) 1-form with values in the adjoint bundle.

There are two things we need to check: (i) $\hat{\zeta}$ is a (nonmanifest) symmetry of L; and (ii) the Lie bracket of transformations (5.32) is consistent with (3.9). We will find the conditions under which (i) and (ii) hold.

First, we investigate $\mathrm{Lie}(\hat{\zeta})L$. The variation of the curvature is

$$\hat{\zeta}F_{\mu\nu} = \eta^a g_{\nu\sigma}\Gamma^\sigma_{ab}D_\mu\lambda^b - \eta^a g_{\mu\sigma}\Gamma^\sigma_{ab}D_\nu\lambda^b,$$

from which we easily compute

$$\hat{\zeta}\big(-\frac{1}{4}g^{\mu\sigma}g^{\nu\rho}\langle F_{\mu\nu}, F_{\sigma\rho}\rangle\big) = -\eta^a g^{\mu\sigma}\Gamma^\nu_{ab}\langle F_{\mu\nu}, D_\sigma\lambda^b\rangle. \tag{5.33}$$

In the variation of the second term in (5.31) we must remember to include the variation of the covariant derivative $D_\sigma = \partial_\sigma + A_\sigma$. Also, since we only want to check whether $\hat{\zeta}$ is a *nonmanifest* symmetry, we are free to compute up to exact terms—that is, to integrate by parts. (To compute the Noether current we would keep track of the exact terms.) We write '$\sim$' to mean *equal up to exact terms*. Then

$$\begin{aligned}\hat{\zeta}\big(\frac{1}{2}\Gamma^\sigma_{ab}\langle\lambda^a, D_\sigma\lambda^b\rangle\big) &\sim \frac{1}{2}\Gamma^\sigma_{ab}\langle\eta^c\tilde{\Gamma}^{\mu ad}\Gamma^\nu_{dc}F_{\mu\nu}, D_\sigma\lambda^b\rangle + \frac{1}{2}\Gamma^\sigma_{ab}\langle\lambda^a, \eta^c g_{\sigma\nu}\Gamma^\nu_{cd}[\lambda^d, \lambda^b]\rangle \\ &\sim \frac{1}{2}\eta^a\Gamma^\sigma_{bd}\tilde{\Gamma}^{\mu cd}\Gamma^\nu_{ac}\langle F_{\mu\nu}, D_\sigma\lambda^b\rangle - \frac{1}{2}\eta^c g_{\sigma\nu}\Gamma^\sigma_{ab}\Gamma^\nu_{cd}\langle\lambda^a, [\lambda^d, \lambda^b]\rangle \\ &\sim \eta^a g^{\mu\sigma}\Gamma^\nu_{ab}\langle F_{\mu\nu}, D_\sigma\lambda^b\rangle - \frac{1}{2}\eta^c g_{\sigma\nu}\Gamma^\sigma_{ab}\Gamma^\nu_{cd}\langle\lambda^a, [\lambda^d, \lambda^b]\rangle.\end{aligned}$$

In the last step we use the Clifford relation $\Gamma^\mu_{bd}\tilde{\Gamma}^{\sigma cd} + \Gamma^\sigma_{bd}\tilde{\Gamma}^{\mu cd} = 2g^{\mu\sigma}\delta^c_b$ and the Bianchi identity $D_\sigma F_{\mu\nu} + D_\nu F_{\sigma\mu} + D_\mu F_{\nu\sigma} = 0$. The first term in the last line cancels against (5.33). Let

$$Q_{abcd} = g_{\sigma\nu}\big(\Gamma^\sigma_{ab}\Gamma^\nu_{cd} + \Gamma^\sigma_{ad}\Gamma^\nu_{bc} + \Gamma^\sigma_{ac}\Gamma^\nu_{db}\big).$$

Then the form of the second term implies:

- If the gauge algebra is abelian, then $\hat{\zeta}$ is a symmetry of L in any dimension and with any spin representation S.
- If the gauge algebra is nonabelian, then $\hat{\zeta}$ is a symmetry of L if and only if the quartic form

$$s^* \longmapsto |\,\Gamma(s^*, s^*)\,|^2, \qquad s^* \in S^*, \tag{5.34}$$

vanishes identically.

We analyze this last condition below.

Next, we investigate the Lie algebra generated by the symmetry transformation (5.32). Let η_1^a, η_2^b be odd parameters and $\hat{\zeta}_i$ the action (5.32) of $\eta_i^a Q_a$. In the abstract Lie algebra (3.9) we have

$$[\eta_1^a Q_a, \eta_2^b Q_b] = -\eta_1^a \eta_2^b [Q_a, Q_b] = 2\eta_1^a \eta_2^b \Gamma_{ab}^\sigma \partial_\sigma.$$

When acting on the spinor field λ, we replace ∂_σ with the covariant derivative D_σ. When acting on the connection A, we use Lie derivative by the horizontal lift, and it is easy to compute that the action of ∂_σ on A has μ^{th} component equal to $F_{\sigma\mu}$. Recall from Lecture 2 that since we use *left* actions of symmetries, the bracket on the space of fields is opposite to the bracket in the abstract supersymmetry algebra. But recall too that the element ∂_σ in the abstract supersymmetry algebra acts by *minus* the derivative on fields. These two minus signs cancel, so we need to check

$$[\hat{\zeta}_1, \hat{\zeta}_2] f = 2\eta_1^a \eta_2^b \Gamma_{ab}^\sigma \partial_\sigma f \tag{5.35}$$

for any field f.

We first verify (5.35) for the connection field. Using (5.32) we find:

$$\begin{aligned}
\left([\hat{\zeta}_1, \hat{\zeta}_2] A\right)_\mu &= -\frac{1}{2}\eta_1^a \eta_2^b g_{\mu\nu} (\tilde{\Gamma}^{\rho cd} \Gamma_{bc}^\nu \Gamma_{ad}^\sigma + \tilde{\Gamma}^{\rho cd} \Gamma_{ac}^\nu \Gamma_{bd}^\sigma) F_{\rho\sigma} \\
&= -\frac{1}{2}\eta_1^a \eta_2^b g_{\mu\nu} (4 g^{\rho\nu} \Gamma_{ab}^\sigma) F_{\rho\sigma} \\
&= 2\eta_1^a \eta_2^b \Gamma_{ab}^\sigma F_{\sigma\mu}.
\end{aligned}$$

To pass to the second line we added to the first line terms in parentheses with ρ and ν exchanged; then we used the Clifford identity (3.3). This is the desired result. To check the commutation relation (5.35) on the spinor field we need to also impose the equation of motion

$$\Gamma_{ab}^\sigma D_\sigma \lambda^a = 0, \qquad \text{for all } b. \tag{5.36}$$

(Recall Example 3 in Lecture 2, where the $U(n)$ symmetry of n coupled harmonic oscillators only "closes" on-shell. The same is true here as well.) Then we compute from (5.32)

(5.37)

$$\begin{aligned}
[\hat{\zeta}_1, \hat{\zeta}_2] \lambda^c = &-\frac{1}{2}\eta_1^a \eta_2^b g_{\mu\nu} (\tilde{\Gamma}^{\sigma cd} \Gamma_{db}^\nu \Gamma_{ae}^\mu + \tilde{\Gamma}^{\sigma cd} \Gamma_{da}^\nu \Gamma_{be}^\mu \\
&\qquad - \tilde{\Gamma}^{\nu cd} \Gamma_{db}^\sigma \Gamma_{ae}^\mu - \tilde{\Gamma}^{\nu cd} \Gamma_{da}^\sigma \Gamma_{be}^\mu) D_\sigma \lambda^e \\
= &\frac{1}{2}\eta_1^a \eta_2^b g_{\mu\nu} \tilde{\Gamma}^{\sigma cd} \Gamma_{de}^\nu \Gamma_{ab}^\mu D_\sigma \lambda^e - \frac{1}{2}\eta_1^a \eta_2^b \tilde{\Gamma}^{\sigma cd} Q_{abde} D_\sigma \lambda^e \\
&\qquad + \frac{1}{2}\eta_1^a \eta_2^b g_{\mu\nu} (\tilde{\Gamma}^{\nu cd} \Gamma_{db}^\sigma \Gamma_{ae}^\mu + \tilde{\Gamma}^{\nu cd} \Gamma_{da}^\sigma \Gamma_{be}^\mu) D_\sigma \lambda^e.
\end{aligned}$$

Let 'LT' denote the last term. Applying the Clifford identity we see that the symmetric part in ν, σ vanishes by the equation of motion, whence

$$\begin{aligned}
LT &= \frac{1}{2} LT - \frac{1}{4}\eta_1^a \eta_2^b g_{\mu\nu} (\tilde{\Gamma}^{\sigma cd} \Gamma_{db}^\nu \Gamma_{ae}^\mu + \tilde{\Gamma}^{\sigma cd} \Gamma_{da}^\nu \Gamma_{be}^\mu) D_\sigma \lambda^e \\
&= \frac{1}{2} LT + \frac{1}{4}\eta_1^a \eta_2^b g_{\mu\nu} \tilde{\Gamma}^{\sigma cd} \Gamma_{de}^\nu \Gamma_{ab}^\mu D_\sigma \lambda^e - \frac{1}{4}\eta_1^a \eta_2^b \tilde{\Gamma}^{\sigma cd} Q_{abde} D_\sigma \lambda^e.
\end{aligned}$$

Solving for LT and plugging into (5.37) we have

$$\begin{aligned}[\hat{\zeta}_1, \hat{\zeta}_2]\lambda^c &= \eta_1^a \eta_2^b g_{\mu\nu} \tilde{\Gamma}^{\sigma cd} \Gamma^{\nu}_{de} \Gamma^{\mu}_{ab} D_\sigma \lambda^e - \eta_1^a \eta_2^b \tilde{\Gamma}^{\sigma cd} Q_{abde} D_\sigma \lambda^e \\ &= 2\eta_1^a \eta_2^b \Gamma^{\sigma}_{ab} D_\sigma \lambda^c - \eta_1^a \eta_2^b \tilde{\Gamma}^{\sigma cd} Q_{abde} D_\sigma \lambda^e.\end{aligned}$$

If the second term vanishes, then this is the desired bracket. Hence:

- The supersymmetry algebra is satisfied on-shell if and only if the quartic form (5.34) vanishes. This applies both to the abelian and nonabelian theories.
- Notice that the abelian theory is *always* invariant under $\hat{\zeta}$, but unless (5.34) vanishes the algebra generated by $\hat{\zeta}$ and the infinitesimal Poincaré transformations is *not* the usual supersymmetry algebra. This does not violate the Haag-Lopuszański-Sohnius theorem, since the abelian theory is free, whereas that theorem gives a restriction on the symmetry algebra of theories in which the S-matrix is nondegenerate.

Finally, we investigate conditions under which the quartic form (5.34) vanishes. From our initial discussion of representation theory we know that this happens in at most the four cases $n = 3, 4, 6, 10$ with minimal supersymmetry $s = 2, 4, 8, 16$, as well as in $n = 2$ with chiral supersymmetry $s = (s^+, 0)$ or $s = (0, s^-)$. For if (5.34) vanishes, then the abelian theory does have the usual supersymmetry algebra, and so the one-particle Hilbert space is a representation of that algebra. In the chiral two-dimensional cases the pairing Γ maps onto one ray in the lightcone, and so the quartic form vanishes. We claim that in the other four cases mentioned, there is in fact no nonzero invariant form of *any* degree on S^*, much less a nonzero invariant quartic form. For $n = 3, 4, 6$ we identify the spin group $\mathrm{Spin}(1, n-1)$ with $SL(2, \mathbb{F})$, $\mathbb{F} = \mathbb{R}, \mathbb{C}, \mathbb{H}$, and we identify S^* with $\mathbb{F}^2$. Since $SL(2, \mathbb{F})$ has the open dense orbit $\mathbb{F}^2 \setminus \{0\} \subset \mathbb{F}^2$, there are no nonzero invariant polynomials on $\mathbb{F}^2$. For the remaining case $n = 10$, the 16-dimensional real half-spin representation S^* of $\mathrm{Spin}(1, 9)$ restricts to the minimal real spin representation of $\mathrm{Spin}(9) \subset \mathrm{Spin}(1, 9)$, and since $\mathrm{Spin}(9)$ is compact there is an invariant metric. The special fact we need is that $\mathrm{Spin}(9)$ acts transitively on the unit sphere in S^*. (This fact is part of the story concerning holonomy groups of irreducible Riemannian manifolds.) Thus the only polynomials on S^* invariant under $\mathrm{Spin}(9)$ are polynomials in the metric. But $\mathrm{Spin}(1, 9)$ does not preserve the metric, so there are no nonzero invariant polynomials at all under $\mathrm{Spin}(1, 9)$. Therefore, in these four cases (5.34) vanishes as claimed.

This completes the proof that (5.30) is supersymmetric.

REFERENCES

The list of references compiled here is hardly exhaustive. Our modest goal is to give the reader an entry into the literature. We make no effort to reference original papers.

There is now a collection of material on quantum field theory aimed specifically at mathematicians:

P. Deligne, P. Etingof, D. S. Freed, L. C. Jeffrey, D. Kazhdan, J. W. Morgan, D. R. Morrison, E. Witten (eds.), *Quantum Fields and Strings: A Course for Mathematicians*, 2 volumes, American Mathematical Society, Providence, RI, 1999.

In particular, Part 1 is a mathematical text which contains more detail about many topics discussed in these lectures. We give specific references to this collection below.

Some standard physics books on supersymmetry (alphabetical by author) are:

P. G. O. Freund, *Introduction to supersymmetry*, Cambridge Monographs on Mathematical Physics, Cambridge University Press, Cambridge, 1986.

S. J. Gates, M. T. Grisaru, M. Roček, W. Siegel, *Superspace, One thousand and one lessons in supersymmetry*, Frontiers in Physics, vol. 58, Benjamin/Cummings Publishing Co. Inc., Reading, Mass., 1983.

M. F. Sohnius, *Introducing supersymmetry*, Phys. Rep. **128** (1985), 39–204.

J. Wess, J. Bagger, *Supersymmetry and supergravity*, Princeton Series in Physics, Princeton University Press, Princeton, NJ, 1992.

P. West, *Introduction to supersymmetry and supergravity*, World Scientific Publishing Co., Inc., Teaneck, NJ, 1990.

We give more specialized references for each individual lecture.

Lecture 1

The ideas of superalgebra and supermanifolds as we have presented them here are found in:

P. Deligne, J. W. Morgan, *Notes on supersymmetry (following J. Bernstein)*, Quantum Fields and Strings: A Course for Mathematicians, *loc. cit.*, pp. 41–97.

This follows the algebro-geometric approach to supermanifolds, which appears in:

D. A. Leites, *Introduction to the theory of supermanifolds*, Uspekhi Mat. Nauk **35** (1980), no. 1, 3–57, translated in Russian Math. Surveys, **35** no. 1 (1980) 1–64.

Yu. I. Manin, *Gauge field theory and complex geometry*, Grundlehren **289**, Second Edition, with an appendix by Sergei Merkulov, Springer-Verlag, Berlin, 1997, pp. 346.

The basic idea is also summarized in:

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There are other mathematical approaches to supermanifolds, which the reader may find discussed in:

A. Rogers, *A global theory of supermanifolds*, J. Math. Phys. **21** (1980), 1352–1365.

B. DeWitt, *Supermanifolds*, 2nd edition, Cambridge University Press, 1992.

M. Batchelor, *Two approaches to supermanifolds*, Trans. Amer. Math. Soc. **258** (1980), 257–270.

M. Rothstein, *The axioms of supermanifolds and a new structure arising from them*, Trans. Amer. Math. Soc. **287** (1986), 159–180.

Lecture 2

The account of classical field theory presented here is based on:

P. Deligne, D. S. Freed, *Classical field theory*, Quantum Fields and Strings: A Course for Mathematicians, *loc. cit.*, pp. 137–225.

An earlier mathematical account is given in:

G. J. Zuckerman, *Action principles and global geometry*, Mathematical Aspects of String Theory, ed. S. T. Yau, World Scientific Publishing, 1987, pp. 259–284.

For Takens' theorem on the cohomology of local forms, see:

F. Takens, *A global version of the inverse problem of the calculus of variations*, J. Diff. Geom. **14** (1979), 543 562.

The argument deriving the Atiyah-Singer index formula for Dirac operators from supersymmetric quantum mechanics appears in:

E. Witten, *Index of Dirac operators*, Notes by J. Morgan, Quantum Fields and Strings: A Course for Mathematicians, *loc. cit.*, pp. 475 511.

Sign conventions are summarized in:

P. Deligne, D. S. Freed, *Sign manifesto*, Quantum Fields and Strings: A Course for Mathematicians, *loc. cit.*, pp. 357–363.

Lecture 3

For details about spinors, especially in Minkowski space, see:

P. Deligne, *Notes on spinors*, Quantum Fields and Strings: A Course for Mathematicians, *loc. cit.*, pp. 99–135.

Our account of classical supersymmetric field theory, in this lectures and in the remaining lectures, is based on:

P. Deligne, D. S. Freed, *Supersolutions*, Quantum Fields and Strings: A Course for Mathematicians, *loc. cit.*, pp. 227–355.

Lecture 4

The supersymmetric σ-model in 2 dimensions is the starting point of:

E. Witten, *Supersymmetry and Morse theory*, J. Diff. Geom. **17** (1982), 661–692.

For a beautiful discussion of solitons in field theory, see:

S. Coleman, *Aspects Of Symmetry*, Cambridge University Press, 1985.

Lecture 5

The classical models described in this lecture are the starting point for the quantum models. The dynamics of quantum theories with 4 supersymmetries in 4 dimensions is recounted in:

N. Seiberg, *Dynamics of $N = 1$ supersymmetric field theories in four dimensions*, Notes by S. Wu, Quantum Fields and Strings: A Course for Mathematicians, *loc. cit.*, pp. 1425–1495.

Quantum theories in 4 dimensions with 8 supersymmetries have applications to four dimensional topology. This, and many other aspects of field theory, are explained in:

E. Witten, *Dynamics of quantum field theory*, Notes by P. Etingof, D. Freed, L. Jeffrey, D. Kazhdan, J. Morgan, D. Morrison, Quantum Fields and Strings: A Course for Mathematicians, *loc. cit.*, pp. 1425–1495.

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